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RELAXATION METHODS APPLIED TO ENGINEERING PROBLEMS

XIV. PLASTIC STRAINING IN TWO-DIMENSIONAL STRESS-SYSTEMS

By D. N. de G. ALLEN and Sir RICHARD SOUTHWELL, F.R.S.

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Only experiment can decide the criterion of elastic failure, and the relation of stress to strain during plastic distortion, in real materials such as steel; and (since stress is not directly measurable) it can do this only by verifying relations deduced from theory in regard to *total* displacements and *resultant* actions. Consequently practical value attaches to computational methods whereby, on the basis of some assumed criterion, relations of that kind can be formulated.

This paper draws conclusions regarding two-dimensional systems (plane stress and plane strain) from the 'Mises-Hencky hypothesis', according to which failure occurs when

$$(p_2-p_3)^2+(p_3-p_1)^2+(p_1-p_2)^2=\text{const.}$$

(p_1, p_2, p_3 denoting the principal stresses), and from the relation

$$\Delta\gamma_1 : \Delta\gamma_2 : \Delta\gamma_3 = q_1 : q_2 : q_3$$

assumed to hold during the subsequent plastic distortion ($\Delta\gamma_1, \Delta\gamma_2, \Delta\gamma_3$ denoting the *incremental* plastic shear-strains and q_1, q_2, q_3 the principal shear stresses). Its methods could be applied to other hypotheses.

In its worked examples some regions remain elastic while in other regions (here termed *enclaves*) the strain is partly plastic. Such cases present special difficulty in an orthodox treatment.

Introduction

1. This paper, like Parts IX and XII, applies the 'method of systematic relaxation of constraints' to a class of problems which for orthodox analysis has great difficulty in that (since the governing equations are not linear) solutions may not be superposed. It is concerned with consequences of a particular hypothesis regarding plastic strain.

The hypothesis is one of many that have been propounded and between which (in relation to actual materials) experiment has yet to decide.* But the difficulties which preclude direct experimental study of stress-strain relations in the elastic range are—to say the least—not reduced by the occurrence of plastic flow. All that can be done is to verify relations deduced from theory in regard to *total* displacements and to *resultant* actions; so computational methods of deriving such relations may be said to have a practical value.

I. Basic theory

The physical hypothesis

2. The torsion problem in relation to material having a finite limit of proportionality was discussed in Part III of this series (Christopherson & Southwell 1938). The diagram relating shear-stress and shear-strain was assumed to have the form of figure 1: under low stress the hypothetical material behaves elastically, but under shear-stress having an

* Recently G. I. Taylor (1947) has suggested a criterion of yield according closely but not exactly with the Mises-Hencky criterion.

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intensity f_Y it was postulated that the shear-strain can take any value in excess of a 'limiting elastic shear-strain' γ_Y. In the torsion problem this hypothesis sets a limit on the gradient of the wanted function, and the restriction can be represented very simply by a device known as the 'Prandtl roof' (Southwell 1946, §§183–189). Here, since we have to deal with complex stress-systems, we shall require a generalization of the hypothesis underlying figure 1. It must contemplate three unequal principal stresses (equivalent to hydrostatic stress combined with three simple shears).

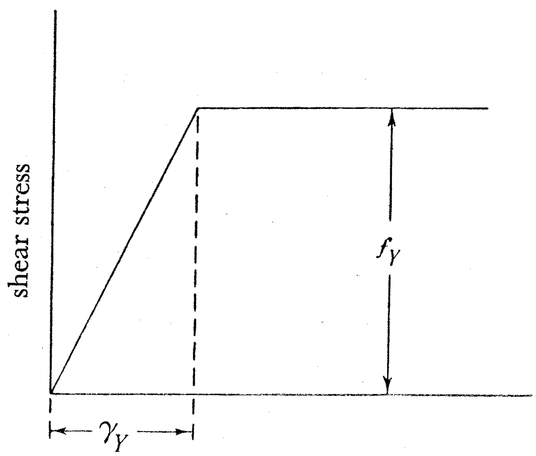

FIGURE 1

3. In the generalization which is adopted here:*

(i) *the first occurrence of plastic straining is determined by the 'Mises-Hencky criterion'*

That is to say, the normal (elastic) relations hold until the principal stresses p_1, p_2, p_3 attain values such that

$$(p_2-p_3)^2+(p_3-p_1)^2+(p_1-p_2)^2 = 8k^2, \tag{1}$$

k being constant for any definite material; and thereafter (in a region of plastic strain) the relation (1) is still satisfied. (Work-hardening is not contemplated.)

(ii) *where the material has yielded plastically, the strain has both an elastic and a plastic part*

The elastic part is related with the stresses by the usual equations

$$(\mathrm{e}_1)_E = \frac{1}{E}\{p_1-\sigma(p_2+p_3)\}, \tag{2}$$

...etc.,

in which E (Young's modulus) and σ (Poisson's ratio) have normal values. The plastic part is *equivoluminal*, i.e.

$$(\mathbf{e}_1)_P+(\mathbf{e}_2)_P+(\mathbf{e}_3)_P = 0, \tag{3}$$

and in it the incremental shear-strains resulting from the three principal shear-stresses are directly proportional to those stresses and act on the same three planes. That is to say, if q_1, q_2, q_3 denote the principal shear-stresses and $\Delta\boldsymbol{\gamma}_1, \Delta\boldsymbol{\gamma}_2, \Delta\boldsymbol{\gamma}_3$ the principal incremental plastic shear-strains,† then

$$\Delta\boldsymbol{\gamma}_1 : \Delta\boldsymbol{\gamma}_2 : \Delta\boldsymbol{\gamma}_3 = q_1 : q_2 : q_3. \tag{4}$$

* Stress-strain relations of 'incremental' type were discussed by L. Prandtl (1924) and by A. Reuss (1930). The latter writes $2k^2$ for the constant denoted by $8k^2$ in (1).

† Hereafter distinct founts, as in (2) and (3), will be employed to distinguish elastic from plastic strains, but the suffixes E and P will be suppressed.

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If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ denote the principal incremental plastic strains, a statement equivalent to (4) is that

$$(\Delta\mathbf{e}_2-\Delta\mathbf{e}_3):(\Delta\mathbf{e}_3-\Delta\mathbf{e}_1):(\Delta\mathbf{e}_1-\Delta\mathbf{e}_2) = (p_2-p_3):(p_3-p_1):(p_1-p_2) \tag{5}$$

and $\Delta\mathbf{e}_1, \Delta\mathbf{e}_2, \Delta\mathbf{e}_3$ *have the directions of* p_1, p_2, p_3. It may be remarked in relation to (4) that

$$\Delta\gamma_1+\Delta\gamma_2+\Delta\gamma_3 = 0, \qquad q_1+q_2+q_3 = 0, \tag{6}$$

independently of any assumption; i.e. the quantities on the left and right of (4) sum to zero, as do those of (5).

4. The assumption (i) is consistent with the assumption made in §2. For there one principal stress was zero and the other two (being equivalent to a pure shear-stress q) had equal and opposite values; that is to say,

$$p_3 = 0, \quad -p_2 = p_1 = q \text{ (say)},$$

and hence, according to (1), $\qquad 6p_1^2 = 8k^2.$

This means that in a region of plastic straining

$$q = p_1 = 2k/\surd 3 \text{ (constant)}. \tag{7}$$

5. In relation to the assumption (ii), §3, a statement is needed of the meaning attached to 'incremental' strains.* It can be made by reference to motion opposed by solid friction—slipping that has close similarity with the 'internal slip' which results in plastic strain. A weight made to slide slowly on a horizontal plane moves at any instant in the direction of the force then acting on it, so its *incremental* displacement may be said to have the direction of the force; but clearly, if that direction is not constant, the same will not normally be true of its *total* displacement measured from some fixed point. To postulate a relation between total displacement and instantaneous force would be (in effect) to endow the weight with an enduring 'memory' of earlier slips; and similarly, to postulate a relation between total strains and instantaneous stresses would be to postulate a 'memory' in over-strained material. It is the 'incremental' principal strains that must (in an isotropic material) have the directions of the principal stresses, in cases where these alter as the plastic region extends.

When they do not alter, neither do the directions of principal strain, and hence, in (4) and (5), total may be substituted for incremental strains. Then an equivalent form of (4) is

$$\gamma_1/q_1 = \gamma_2/q_2 = \gamma_3/q_3 = 3\lambda \text{ (say)}, \tag{8}$$

$\gamma_1, \gamma_2, \gamma_3$ denoting the *total* plastic shear-strains: that is, the shear stress-strain relations have the same forms in plastic as in elastic straining—the only difference being that λ, *in* (8), *can have any positive value*.† ($\lambda = 0$ when the strain is wholly elastic.)

6. In general the directions of principal stress are *not* invariant. Then (4) may be replaced by

$$\Delta\gamma_1/q_1 = \Delta\gamma_2/q_2 = \Delta\gamma_3/q_3 = 3\Delta\lambda, \tag{4A}$$

and (5) by

$$\frac{(\Delta\mathbf{e}_2-\Delta\mathbf{e}_3)}{p_2-p_3} = \frac{(\Delta\mathbf{e}_3-\Delta\mathbf{e}_1)}{p_3-p_1} = \frac{(\Delta\mathbf{e}_1-\Delta\mathbf{e}_2)}{p_1-p_2} = \tfrac{3}{2}\Delta\lambda, \tag{5A}$$

* Throughout this paper Δ's distinguish 'incremental' quantities.

† The stresses had invariant directions in the plastic-torsional solution of Part III (cf. §2 and Appendix).

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$\Delta\lambda$ denoting an incremental quantity of the type of λ. So

$$2\Delta\mathbf{e}_1-\Delta\mathbf{e}_2-\Delta\mathbf{e}_3=\tfrac{3}{2}\Delta\lambda(2p_1-p_2-p_3),$$

and hence, according to (3),

$$\Delta\mathbf{e}_1=\Delta\lambda\{p_1-\tfrac{1}{2}(p_2+p_3)\},\ \ldots\text{ etc.} \tag{9}$$

That is to say, *as regards the plastic part of the distortion the (stress)-(incremental strain) relations have normal forms, the effective values of Young's modulus and of Poisson's ratio being* $1/\Delta\lambda$ *and* $\tfrac{1}{2}$ *respectively.*

The total strains associated with given stresses are, according to § 3 (ii), the sums of these plastic and of normal elastic components, so that (e.g.) the total strain

$$e_1=\mathsf{e}_1+\mathbf{e}_1;$$

and the fact that (2) relates to total, (9) to *incremental* strains, both expressible in terms of *total* stresses, entails a serious complication of the formal problem. Combined, (2) and (9) yield three equations of the type

$$\Delta e_1=\frac{1}{E}\{\Delta p_1-\sigma(\Delta p_2+\Delta p_3)\}+\Delta\lambda\{p_1-\tfrac{1}{2}(p_2+p_3)\}. \tag{10}$$

These are the basis of our solutions.

Systems of plane strain and of plane stress. (1) *The criterion of plasticity*

7. This paper, being concerned with systems of plane stress and of plane strain, will utilize the two-dimensional equations of transformation for stress and strain. Let z be the direction of the principal stress p_3, and let the material be isotropic so that planes perpendicular to z are principal planes both of stress and strain. Then, if x', y' and x, y (both perpendicular to z) are related by the scheme

	x	y
x'	$\cos\theta$	$\sin\theta$
y'	$-\sin\theta$	$\cos\theta$

, (11)

the equations of transformation for stress are

$$\left.\begin{aligned}X'_{x'}&=X_x\cos^2\theta+Y_y\sin^2\theta+X_y\sin 2\theta,\\ Y'_{y'}&=X_x\sin^2\theta+Y_y\cos^2\theta-X_y\sin 2\theta,\\ Z'_{z'}&=Z_z=p_3,\quad Y'_{z'}=Z'_{x'}=0,\\ X'_{y'}&=-\tfrac{1}{2}(X_x-Y_y)\sin 2\theta+X_y\cos 2\theta,\end{aligned}\right\} \tag{i}$$

and the equations of transformation for strain are

$$\left.\begin{aligned}e_{x'x'}&=e_{xx}\cos^2\theta+e_{yy}\sin^2\theta+\tfrac{1}{2}e_{xy}\sin 2\theta,\\ e_{y'y'}&=e_{xx}\sin^2\theta+e_{yy}\cos^2\theta-\tfrac{1}{2}e_{xy}\sin 2\theta,\\ e_{z'z'}&=e_{zz},\quad e_{y'z'}=e_{z'x'}=0,\\ e_{x'y'}&=-(e_{xx}-e_{yy})\sin 2\theta+e_{xy}\cos 2\theta.\end{aligned}\right\} \tag{ii}$$

The other two planes of principal stress have directions given by those values of θ for which $X'_{y'}=0$, i.e. for which

$$\tan 2\theta=2X_y/(X_x-Y_y), \tag{12}$$

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and accordingly are perpendicular. The magnitudes of the principal stresses are, by (i) and (12),

$$p_1, p_2 = \tfrac{1}{2}(X_x + X_y) \pm \tfrac{1}{2}\{(X_x - Y_y)^2 + 4X_y^2\}^{\frac{1}{2}}. \tag{13}$$

The planes on which $X'_{y'}$ has stationary values have directions given by those values of θ for which $\frac{\partial}{\partial\theta}X'_{y'} = 0$, i.e. for which

$$-\cot 2\theta = \frac{2X_y}{X_x - Y_y}, \tag{14}$$

and so are also perpendicular. By (12) and (14) they are shown to be inclined at 45° to (that is, to bisect the angles between) the planes of principal stress. On them, by (i) and (14), the shear-stress has intensity $\pm q$, where

$$4q^2 = (X_y - Y_x)^2 + 4X_y^2,$$

and the normal stress has intensity

$$\tfrac{1}{2}(X_x + Y_y). \tag{15}$$

8. When p_1, p_2 are given their expressions (13), the Mises-Hencky criterion (1) requires that

$$(X_x + Y_y - 2p_3)^2 + 3\{(X_x - Y_y)^2 + 4X_y^2\} \leqslant 16k^2, \tag{16}$$

the sign of inequality relating to elastic conditions, and the sign of equality to conditions of plastic straining.

In *plane stress* p_3 is zero everywhere, so the criterion (16) reduces to

$$(X_x + Y_y)^2 + 3\{(X_x - Y_y)^2 + 4X_y^2\} \leqslant 16k^2,$$

i.e. to

$$X_x^2 - X_x Y_y + Y_y^2 + 3X_y^2 \leqslant 4k^2. \tag{17}$$

9. For plane strain too a criterion involving only X_x, X_y and Y_y will be needed in order that those stress-components may be determined; but to derive this from (16) is difficult unless the simplifying assumption is introduced that $\sigma = \frac{1}{2}$ (i.e. that the elastic like the plastic strain is equivoluminal) within a region of plastic straining. For in plane strain e_{zz} is zero everywhere, and hence Δe_{zz}; so, by the third equation of type (10),

$$\frac{1}{E}\{\Delta p_3 - \sigma(\Delta X_x + \Delta Y_y)\} + \tfrac{1}{2}\Delta\lambda(2p_3 - X_x - Y_y) = \Delta e_{zz} = 0. \tag{18}$$

Now initially (while the strain is wholly elastic) evanescence of e_{zz} requires that

$$p_3 = \sigma(X_x + Y_y), \tag{19}$$

so (18) becomes

$$\frac{1}{E}\{\Delta p_3 - \sigma(\Delta X_x + \Delta Y_y)\} - \tfrac{1}{2}(1 - 2\sigma)\,\Delta\lambda(X_x + Y_y) = 0,$$

and (19) cannot also hold after the start of plastic straining (except locally, at points where either $\Delta\lambda$ or $(X_x + Y_y)$ is zero) in material for which $\sigma \neq \frac{1}{2}$. In such material $p_3/(X_x + Y_y)$ may be expected, as plastic strain extends, to change gradually from σ to a value near to $\frac{1}{2}$; thereby entailing a gradual change in the form of (16) regarded as a relation between X_x, X_y and Y_y. This circumstance makes an exact treatment very complicated.

On that account (and since the consequent error may be expected to be small) we shall henceforth neglect the first term $(X_x + Y_y - 2p_3)^2$ in (16)—thereby (in effect) giving σ the value $\frac{1}{2}$—and take as the criterion appropriate to plane strain

$$(X_x - Y_y)^2 + 4X_y^2 \leqslant \frac{16k^2}{3}, \tag{20}$$

whatever value of σ may be appropriate in (10).

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Systems of plane strain and of plane stress. (2) *The computation of stress*

10. Both in plane strain and in plane stress, provided that no body force is operative, all of the conditions of equilibrium are satisfied when

$$X_x = \frac{\partial^2\chi}{\partial y^2}, \quad X_y = -\frac{\partial^2\chi}{\partial x\,\partial y}, \quad Y_y = \frac{\partial^2\chi}{\partial x^2}, \tag{21}$$

whatever be the nature of the accompanying strains; and when (as in this paper) known *tractions* act at a single boundary, χ and $\partial\chi/\partial\nu$ have the same boundary values whether plastic straining has occurred or no. For on the boundary, as was shown in Part VII A (Fox & Southwell 1941), §7,

$$-\frac{\partial\chi}{\partial x} = \int Y_\nu\,ds, \qquad \frac{\partial\chi}{\partial y} = \int X_\nu\,ds, \tag{22}$$

both quantities being single-valued; and the expression

$$\begin{aligned}\chi &= x\frac{\partial\chi}{\partial x} + y\frac{\partial\chi}{\partial y} - \int\left(x\frac{\partial^2\chi}{\partial s\,\partial x} + y\frac{\partial^2\chi}{\partial s\,\partial y}\right)ds \\ &= x\frac{\partial\chi}{\partial x} + y\frac{\partial\chi}{\partial y} + \int (xY_\nu - yX_\nu)\,ds, \quad \text{by (22)},\end{aligned} \tag{23}$$

satisfies both of (22) and makes χ single-valued.* (The relations of x, y, ν and s are as shown in figure 2.)

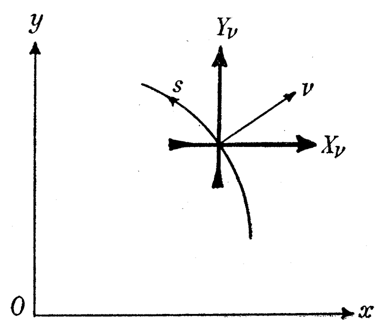

FIGURE 2

11. Edge tractions being specified, boundary values of χ, $\partial\chi/\partial x$ and $\partial\chi/\partial y$ can be computed in accordance with (22) and (23) *and will not be altered by the occurrence of plastic straining*. So long as the whole of the plate remains elastic, computation of χ at internal points is a biharmonic problem, soluble by the methods of Part VII A. Its solution is unique apart from a multiplying factor which (by Hooke's law) increases in proportion to the total load and also attaches to the stresses. Therefore by substitution in the criterion (17) or (20) a value can be found for the greatest load that will be sustained elastically.

If now the load is increased a little further, in some parts the criterion will be violated, and there the elastic solution must be modified so as to satisfy it *while leaving* χ, $\partial\chi/\partial x$ *and* $\partial\chi/\partial y$ *unaltered on the boundary*. The change can be effected by a relaxational procedure explained below (§§25–28). Orthodox treatment is especially difficult in cases where plastic straining is restricted to one or more parts almost or entirely surrounded by material which remains elastic. Such cases have accordingly been made a particular concern of this paper, in which (for brevity) regions of plastic straining are termed *enclaves*.

* Multiply-connected plates are here excluded. They were given special treatment in Part XIII (Southwell 1948).

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As a rule the stress-distribution will be found to alter both within the *enclaves* and—to a less but perceptible extent—in the still elastic material which adjoins them. On that account the planes of maximum shear-stress will change direction as the *enclaves* extend, and for this reason the load must be increased by steps; but in fact the directions are found to alter slowly, so the steps may be fairly large.

Continuity at a plastic-elastic interface

12. At this point it is necessary to consider whether any component displacement, strain or stress can be discontinuous at an interface; for the argument used in the theory of elasticity, that discontinuous displacement would entail an infinite strain and therefore stress, plainly is no longer applicable when (§2) an infinite strain can consist with finite stress.

Continuity of *displacement* is an assumption, necessary to the progress of this investigation, which almost certainly is valid in respect of its examples. (We cannot exclude the physical possibility of finite 'slip' at a plastic-elastic interface, but it would seem to be a reasonable conjecture—frequently made in work in soil mechanics—that when such slip occurs the interface is either straight or circular, and neither is the fact in our solutions.) On that assumption e_{zz} must be continuous, also the stretch *in the direction of the interface*.

13. Continuity of *stress* can be postulated if (on the ground that an interface consists of points at which the conditions of plastic straining are just attained) the quantity on the left of the Mises-Hencky criterion (17) or (20) is assumed to have the same value on both sides of it. For if Ox has the local direction of the interface, it may be postulated that both X_y and Y_y are continuous, since otherwise the conditions of equilibrium would require $\partial X_x/\partial x$ and $\partial X_y/\partial x$ to take infinite values. Then, according to (17),

$$X_x^2 - X_x Y_y + \tfrac{1}{4} Y_y^2 = (X_x - \tfrac{1}{2} Y_y)^2$$

is continuous, therefore $(X_x - \frac{1}{2}Y_y)$ and hence X_x; and a like conclusion may be drawn from (20), which shows $(X_x - Y_y)$ to be continuous.

14. The continuity thus established for stresses will of course hold also in respect of incremental stress-components. Accordingly the elastic parts of Δe_1, Δe_2, Δe_3 as given by (10), §6, must be continuous, and hence the elastic part of every strain-component. Now in §12 the *total* strain e_{zz} was shown to be continuous, also the total strain e_{xx} when (as above) Ox has the local direction of the interface. It follows that the plastic parts of e_{zz} and e_{xx} must (since they are zero on its elastic side) also be zero at the interface; that is to say,

$$\Delta\lambda\{Z_z - \tfrac{1}{2}(X_x + Y_y)\} = 0 = \Delta\lambda\{X_x - \tfrac{1}{2}(Y_y + Z_z)\}.$$

For *plane stress* (since $Z_z = 0$) these relations require $\Delta\lambda$ to vanish at the interface, excepting (possibly) at points where $X_x = Y_y = 0$. For *plane strain* they require $\Delta\lambda$ to vanish at the interface, excepting (possibly) where $X_x = Y_y = Z_z$. But where $X_x = Y_y$ the direction (Ox) of the interface coincides (cf. (14), §7) with a plane of principal shear-stress, and a finite value for $\Delta\lambda$ would imply discontinuity of the principal shear-strain. *Accordingly we now postulate that* $\Delta\lambda = 0$ *at every point on an interface*, remembering that further examination of the solution will be necessary if $X_x = Y_y$ at any such point. *Then at the interface every strain-component is continuous* (its plastic part being zero).

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Plane strain and plane stress in plastically strained material

15. It is also necessary to examine in relation to a plastically strained material the accepted theories of plane strain and of plane stress. Clearly the strain-components will still be subject to the usual conditions of compatibility (cf., e.g., Southwell 1941, §308), because these are purely kinematical. For the same reason they will also govern the incremental strains.

In *plane strain* (Southwell 1941, §§400–402)

$$e_{xz} = e_{yz} = e_{zz} = 0 = \frac{\partial}{\partial z}(e_{xx}, e_{xy}, e_{yy}), \tag{24}$$

so of the six equations of compatibility all are satisfied except one, viz.

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \partial y}. \tag{25}$$

This, when e_{xx}, e_{xy}, e_{yy} are wholly elastic, leads to the biharmonic equation

$$\nabla^4 \chi = 0 \tag{26}$$

when the stresses are given their expressions (21). But when the plastic criterion (17) is operative, χ no longer satisfies equation (26), and hence the elastic parts of Δe_{xx}, Δe_{xy}, Δe_{yy} no longer satisfy a relation of the type of (25). The plastic parts must take such values (in virtue of their undetermined factor $\Delta\lambda$) that a relation of the type of (25) is satisfied by the *total* strain.

Here we confront the difficulty, noticed in §9, of giving a value to the principal stress Z_z (or p_3). The postulated evanescence of e_{zz} would require

$$Z_z = p_3 = \sigma(X_x + Y_y) \tag{19 bis}$$

if e_{zz} were wholly elastic, and

$$Z_z = p_3 = \tfrac{1}{2}(X_x + Y_y) \tag{27}$$

if e_{zz} were wholly due to plastic straining. In §9 we circumvented the difficulty by an assumption which led to a slightly inexact criterion (20). Here (with a similar aim of avoiding computational complexity at a cost in accuracy which may be expected to be small) we shall make two slightly conflicting assumptions: substituting from

$$\Delta Z_z = \sigma(\Delta X_x + \Delta Y_y) \tag{28}$$

in the elastic parts of the expressions for Δe_{xx} and Δe_{yy}, and from

$$Z_z = \tfrac{1}{2}(X_x + Y_y) \tag{29}$$

in the plastic parts.

Then the stress-strain relations in a plastic *enclave* become (cf. §6)

$$\left.\begin{aligned} \Delta e_{xx} &= \frac{1+\sigma}{E}\{(1-\sigma)\,\Delta X_x - \sigma \Delta Y_y\} + \tfrac{3}{4}\Delta\lambda(X_x - Y_y), \\ \Delta e_{yy} &= \frac{1+\sigma}{E}\{(1-\sigma)\,\Delta Y_y - \sigma \Delta X_x\} + \tfrac{3}{4}\Delta\lambda(Y_y - X_x), \\ \Delta e_{xy} &= 2\frac{1+\sigma}{E}\,\Delta X_y + 3\Delta\lambda X_y. \end{aligned}\right\} \tag{30}$$

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16. *Plane stress* calls for somewhat fuller discussion because in general $e_{xx}, e_{xy}, e_{yy}, e_{zz}$ are *not* independent of z, but does not require any simplifying assumption to be made at a cost in exactitude. (This is a fortunate circumstance, for plane stress has more practical importance than plane strain.)

Our one initial assumption is justified by its results. It is, that (as in the *elastic* theory of plane stress) every non-zero strain-component is expressible in the form

$$e = (e) + \tfrac{1}{2}z^2 e', \tag{31}$$

where (e) and e' are functions of x and y but not of z. By definition $X_z = Y_z = Z_z = 0$ everywhere and always, so e_{yz} and e_{zx} may be made zero in the equations of compatibility. Each of those equations, on the assumption stated in (31), will contain a part which involves and a part which is independent of z, and both parts must vanish severally. The parts which are independent of z entail the relations

$$\frac{\partial^2}{\partial x^2}(e_{zz}) + e'_{xx} = 0, \quad \frac{\partial^2}{\partial y^2}(e_{zz}) + e'_{yy} = 0, \quad 2\frac{\partial^2}{\partial x\,\partial y}(e_{zz}) + e'_{xy} = 0, \tag{32}$$

and
$$\frac{\partial^2}{\partial y^2}(e_{xx}) + \frac{\partial^2}{\partial x^2}(e_{yy}) = \frac{\partial^2}{\partial x\,\partial y}(e_{xy}), \tag{33}$$

dashes and brackets having the same significance as in (31). The parts involving z entail the relations*

$$\frac{\partial^2}{\partial x^2}e'_{zz} = \frac{\partial^2}{\partial y^2}e'_{zz} = \frac{\partial^2}{\partial x\,\partial y}e'_{zz} = 0, \tag{34}$$

with
$$2\frac{\partial}{\partial y}e'_{xx} = \frac{\partial}{\partial x}e'_{xy}, \quad 2\frac{\partial}{\partial x}e'_{yy} = \frac{\partial}{\partial y}e'_{xy}. \tag{35}$$

All of (32)–(35) have to be satisfied, and within as well as outside of the plastic *enclave*, independently of any assumption in regard to plastic straining other than what has been stated in (31).

17. From (34) it follows that e'_{zz} is linear in x and y, and outside of the *enclave* it is known (from the theory of elasticity) to be zero; so, for continuity of displacement at the plastic-elastic interface, e'_{zz} must be zero everywhere excepting (possibly) where the interface is straight—a case which we have excluded.

The same is true of $(e'_{xx} + e'_{yy})$—for continuity it must vanish everywhere; and hence, according to the first and second of (32),

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right](e_{zz}) = 0 \tag{36}$$

within as well as outside of the plastic *enclave*.†

* There is a sixth equation involving z; but when (35) are satisfied it reduces to an identity.

† Equations (35) and the last of (32) are compatible with these results. For according to (35) we may write

$$2e'_{xx} = -2e'_{yy} = \frac{\partial \phi}{\partial x}, \quad e'_{xy} = \frac{\partial \phi}{\partial y},$$

where ϕ is plane-harmonic; and then in virtue of (36) all three of the relations (32) are satisfied if

$$\phi + 2\frac{\partial}{\partial x}(e_{zz}) = 0.$$

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Moreover, since $e'_{xx}, e'_{xy}, e'_{yy}$ are continuous at an interface, so also, by (32), are

$$\frac{\partial^2}{\partial s\,\partial x}(e_{zz}) \quad \text{and} \quad \frac{\partial^2}{\partial s\,\partial y}(e_{zz}),$$

and hence (once more excluding the case of a straight interface) not only is (e_{zz}) continuous, but also $\partial(e_{zz})/\partial x$ and $\partial(e_{zz})/\partial y$. It has been shown that $e'_{zz} = 0$ everywhere, so (e_{zz}) may be replaced by e_{zz}, the *total* strain. We conclude (1) that e_{zz} does not vary throughout the thickness of the plate, (2) *that* e_{zz} *is a plane-harmonic function of x and y having complete continuity at the plastic-elastic interface.*

18. The same conclusions hold in respect of Δe_{zz}, which has within the interface the expression

$$\Delta e_{zz} = -\frac{\sigma}{E}(\Delta X_x + \Delta Y_y) - \tfrac{1}{2}\Delta\lambda(X_x + Y_y). \tag{37}$$

If, then, $(\Delta X_x + \Delta Y_y)$ has been determined everywhere, Δe_{zz} is calculable in the elastic region; and within the *enclave*, when Δe_{zz} has been determined, $\Delta\lambda$ can be found from (37). The other strain-components are given by

$$\left.\begin{aligned} \Delta e_{xx} &= \frac{1}{E}(\Delta X_x - \sigma\Delta Y_y) + \Delta\lambda(X_x - \tfrac{1}{2}Y_y), \\ \Delta e_{yy} &= \frac{1}{E}(\Delta Y_y - \sigma\Delta X_x) + \Delta\lambda(Y_y - \tfrac{1}{2}X_x), \\ \Delta e_{xy} &= \frac{2(1+\sigma)\Delta X_y}{E} + 3\Delta\lambda\, X_y. \end{aligned}\right\} \tag{38}$$

Systems of plane strain and of plane stress. (3) *The computation of strain*

19. A relaxational technique for computing stresses is explained in §§25–28. When X_x, X_y, Y_y and their incremental values have been thus determined, the problem remains of computing strains and displacements. The strains are given by (30) under conditions of plane strain, and by (38) under conditions of plane stress. Before they can be evaluated in an *enclave*, the quantity $\Delta\lambda$ must be determined.

Our treatment postulates that the total strains, though large in relation to what can be sustained elastically, are small enough to justify the customary approximations

$$e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},$$

which yield (25), §15, as a condition for compatibility of strain. In an elastic region that condition is satisfied when (the usual relations holding between stress and strain) the stress-components have the expressions (21) in terms of a biharmonic function χ. Within an *enclave* only *total* strains are required to be compatible, so *we cannot speak of displacements as having separable elastic and plastic parts.* It is because the elastic parts of $\Delta e_{xx}, \Delta e_{yy}, \Delta e_{zz}$ are not compatible (cf. §15) that $\Delta\lambda$ must take finite values.

Two methods have been devised for the determination of $\Delta\lambda$, neither involving 'relaxational' processes. The first (due to R.V.S.) on test proved to be practicable but inconvenient by reason of the large amount of interpolation which it entails. (It calls first for a computation,

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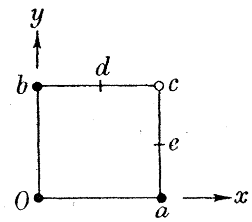
based on (14) of § 7, of intersecting 'trajectories' of maximum shear-stress, and subsequently for integrations along those trajectories to determine the resolved parts of the local displacement.) The second (due to D.N. de G.A.) was found to meet all requirements. It is a process akin to extrapolation, whereby knowledge of u and v is systematically extended from mesh to mesh of the net used in the computation of χ. The essential notion is that (within the accuracy of certain approximations in finite differences) knowledge of the incremental displacements Δu and Δv at any three corners of a mesh, combined with knowledge of X_x, X_y, Y_y at all four corners, permits Δu, Δv and $\Delta\lambda$ to be determined, for the fourth corner, from a condition for the compatibility of the total strains.



20. In the appended diagram let O, a, b, c be corners of a square mesh of side a, and d, e the middle points of bc and ac respectively; and suppose that Δu, Δv have been determined at O, a and b, X_x, X_y and Y_y at all four corners. Such knowledge may always be presumed in respect of *some* mesh taken as a starting point: our purpose now is to extend it, mesh by mesh, to *all* nodal points of the 'ultimate' relaxation net.

It will be convenient to employ new symbols as under for those derivatives of Δu and Δv which our object is to determine for c from values given for O, a and b:

$$\left.\begin{aligned} A &\equiv a\Delta e_{xx} = a\frac{\partial \Delta u}{\partial x}, \quad B \equiv a\Delta e_{yy} = a\frac{\partial \Delta v}{\partial y}, \\ C &\equiv a\Delta e_{xy} = D+E, \quad \text{where } D \equiv a\frac{\partial \Delta u}{\partial y}, \quad E \equiv a\frac{\partial \Delta v}{\partial x}. \end{aligned}\right\} \tag{39}$$

Then, within the accuracy of the usual finite-difference approximations to the derivatives, the relations

$$\left.\begin{aligned} A_b + A_c &= 2A_d = 2(\Delta u_c - \Delta u_b), \quad D_a + D_c = 2D_e = 2(\Delta u_c - \Delta u_a), \\ B_a + B_c &= 2B_e = 2(\Delta v_c - \Delta v_a), \quad E_b + E_c = 2E_d = 2(\Delta v_c - \Delta v_b), \end{aligned}\right\} \tag{40}$$

will hold; and from them, by elimination of Δu_c, Δv_c, there results

$$A_c + B_c - C_c = 2(\Delta u_a - \Delta u_b - \Delta v_a + \Delta v_b) - B_a - A_b + D_a + E_b, \tag{41}$$

in which all quantities on the right are known. From this equation $(\Delta\lambda)_c$ can be determined after substitution for A_c, B_c and C_c on its left-hand side. Under conditions of plane strain, by (39) and (30),

$$(A_c + B_c - C_c)/a = \left[\frac{1+\sigma}{E}\{(1-2\sigma)(\Delta X_x + \Delta Y_y) - 2\Delta X_y\} - 3\Delta\lambda\, X_y\right]_c, \tag{42}$$

and under conditions of plane stress, by (39) and (38),

$$(A_c + B_c - C_c)/a = \left[\frac{1}{E}\{(1-\sigma)(\Delta X_x + \Delta Y_y) - 2(1+\sigma)\Delta X_y\} + \Delta\lambda\{\tfrac{1}{2}(X_x + Y_y) - 3X_y\}\right]_c. \tag{43}$$

Thus in both instances the equation giving $(\Delta\lambda)_c$ is *linear*.

21. Knowing $(\Delta\lambda)_c$ we can give definite values to A_c and B_c, and then the first and second of (40) give Δu_c and Δv_c. Inserting their values in the third and fourth of (40) we can determine

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D_c and E_c; then repeat the whole of the foregoing cycle of operations in relation to an adjoining mesh.

The process though somewhat laborious is simple, and it yields for every nodal point in turn consistent values of Δu, Δv and $\Delta\lambda$. Errors are not carried forward as they are in a normal extrapolation (sometimes with increased magnitudes at every step); instead, they merely entail corresponding small errors in $\Delta\lambda$—which is preferable since these errors can be removed by 'smoothing'.

22. Applied to elastic strains computed by means of (21) from a stress-function which is accurately biharmonic, the process would yield zero values for $\Delta\lambda$ because (§19) such strains are strictly 'compatible'. But χ-values found by a relaxational technique will inevitably contain small errors, revealed by small residuals which have (if the relaxation has been conducted properly) no *systematic* distribution. Such errors, though intrinsically small, will (§21) be magnified in an ordinary extrapolation; whereas by the process here described they are not 'carried forward' but are exhibited locally, by small and unsystematically distributed λ's.* Thereby the errors of a relaxational stress-computation are given a quasi-physical interpretation: *the λ's are measures of small variations in the elastic constants which, if they really existed, would make the solution exact.*

II. The relaxational attack

The worked examples

23. To fix ideas we shall consider more particularly the two examples which have been solved by a relaxational technique—tensile specimens in which the stresses are intensified by notches giving a central 'waist'. They differ only in respect of these notches, which in Example 1 are semicircular, conforming with an example treated by Coker & Filon (1931). Their treatment employed the photo-elastic method and was thus restricted to *elastic* stresses, which can be computed by the methods of Part VII A (Fox & Southwell 1931). Figure 3 gives elastic stresses computed by Miss G. Vaisey, whose contours of 'equivalent stress' (3*d*) agree closely with the 'isochromatics' shown in figure 7·02 of Coker & Filon's book. Both studies suggest that regions of fairly intense stress-difference (likely to entail *enclaves* of plasticity) exist not only near the notches but also on the centre-line at points A, A near the 'waist'.

In Example 2 the notches are sharp re-entrant right angles, and the regions of intense stress-difference have quite different shapes. Figure 4 exhibits results of computation (by D.N. de G.A.) for stresses *within* the elastic range. We do not know of any comparable results of photo-elastic study.

24. Having these 'elastic' solutions we can investigate the modifications which result when, in a region of plastic straining, the condition (17) or (20) supervenes. It appears from figures 3*c* and 3*d* that a plastic *enclave*, starting at the bottom of each notch, will extend into the specimen from those points, and may be accompanied later by two new *enclaves* starting near the points A, A. Within an *enclave* the initial (elastic) distribution of χ has to be modified by relaxation methods in accordance with (17) or (20).

* λ is here used in place of $\Delta\lambda$, because in an elastic region it is not necessary to consider 'incremental' strains.

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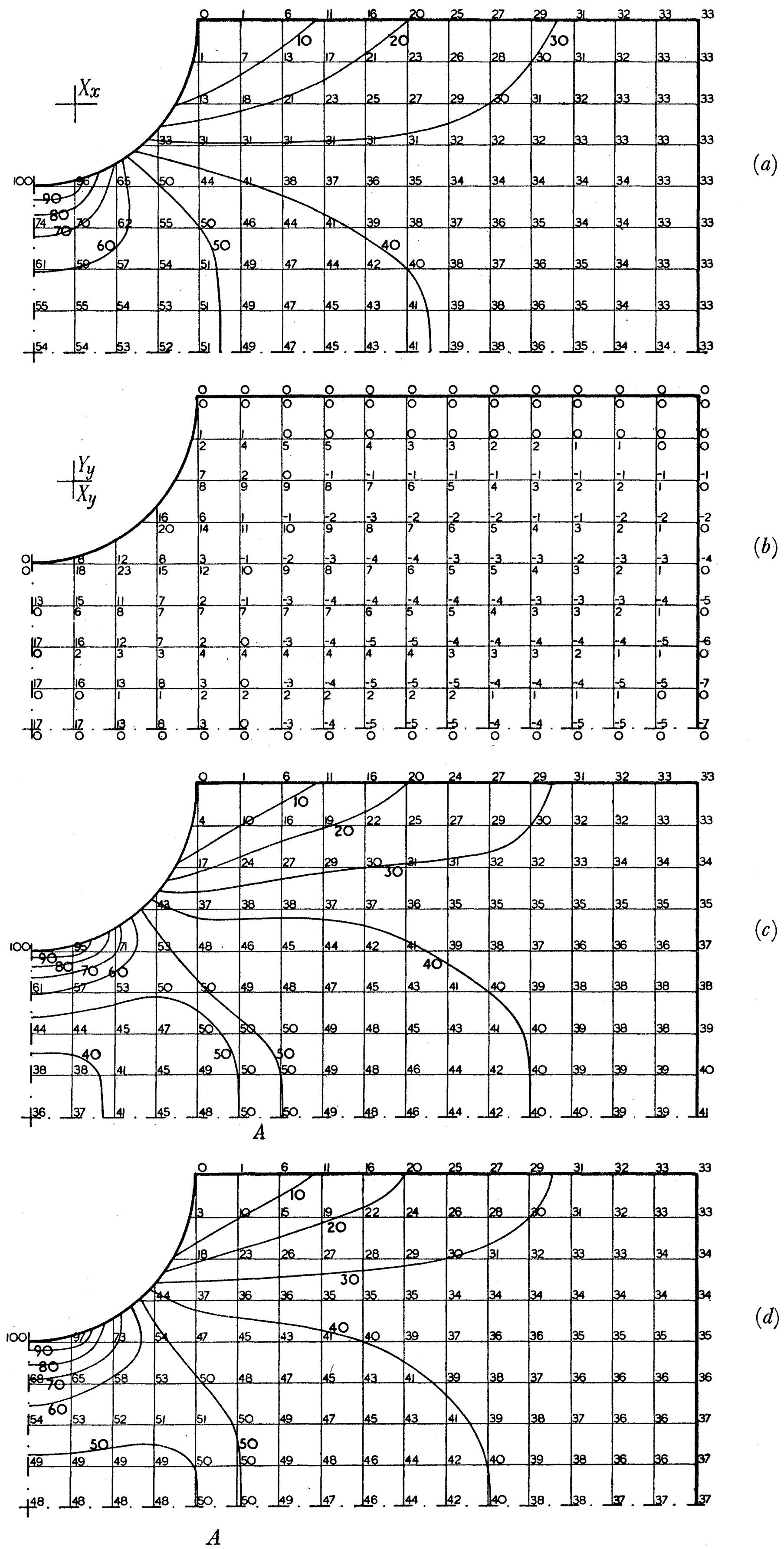

FIGURE 3. Stress-components and 'equivalent stress' (G. Vaisey). (a) X_x. (b) X_y and Y_y. (c) 'Mises-Hencky equivalent stress' (plane strain). (d) 'Mises-Hencky equivalent stress' (plane stress).

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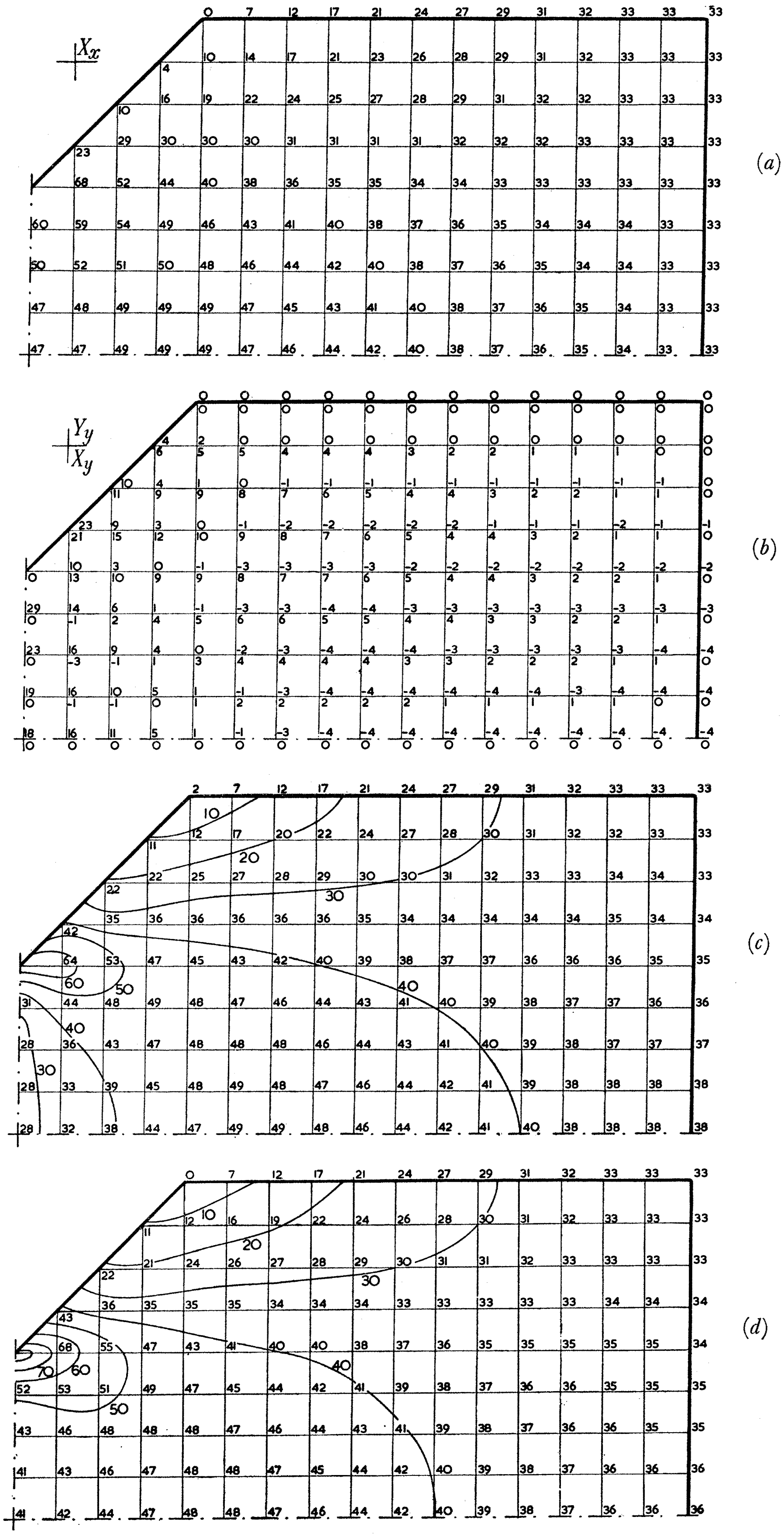

FIGURE 4. Stress-components and 'equivalent stress' (D.N. de G.A.). Lettering (a) to (d) as for figure 3.

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The relaxational computation of stress

25. Let θ, ϕ and ψ be functions defined by

$$\left.\begin{aligned} \theta &= X_x + Y_y = \nabla^2\chi \\ \phi &= -(X_x - Y_y) = \left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right]\chi \\ \psi &= -2X_y = 2\frac{\partial^2\chi}{\partial x\,\partial y} \end{aligned}\right\} \text{ according to (21)} \tag{44}$$

(∇^2 now standing for $\partial^2/\partial x^2 + \partial^2/\partial y^2$), so that *identically*

$$2\frac{\partial^2\phi}{\partial x\,\partial y} = \left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right]\psi. \tag{45}$$

Both (17) and (20) may be written in the form

$$Q^2 \text{ (say)} = \mathbf{a}\theta^2 + \mathbf{b}(\phi^2 + \psi^2) \leqslant \mathbf{c}^2, \tag{46}$$

where **a**, **b**, **c** are constants. In (17)

$$\mathbf{a} = 1, \quad \mathbf{b} = 3, \quad \mathbf{c} = 4k; \tag{47}$$

and in (20)

$$\mathbf{a} = 0, \quad \mathbf{b} = 1, \quad \mathbf{c} = 4k/\surd 3. \tag{48}$$

All of (44) to (46) hold equally in respect of elastic or plastic straining, and both of plane strain and plane stress.

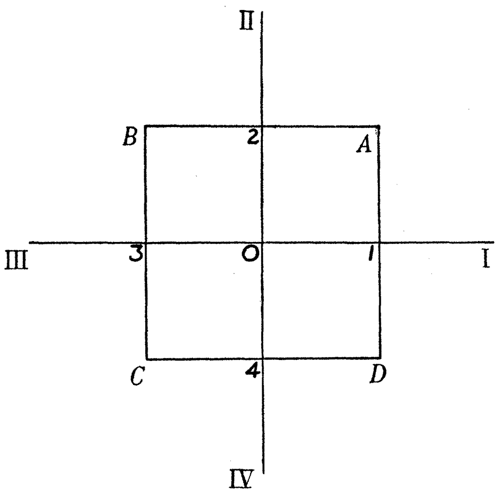

FIGURE 5

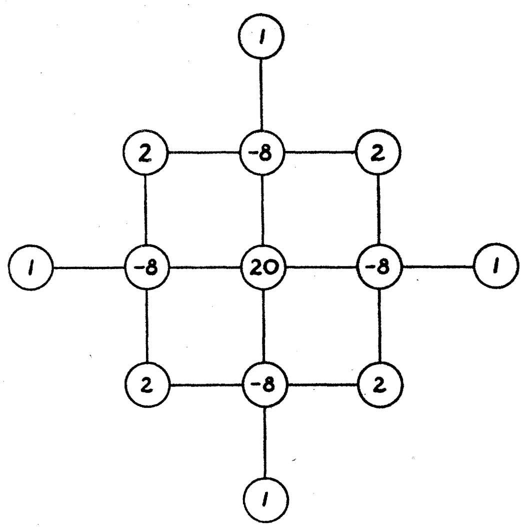

FIGURE 6

26. In the elastic region (no body force being operative)

$$\nabla^4\chi = 0, \tag{26 bis}$$

and so, according to (44),

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right]\phi + 2\frac{\partial^2\psi}{\partial x\,\partial y} = 0. \tag{49}$$

'Residuals' may be defined by reference to a finite-difference approximation to (26), namely,

$$(\mathbf{F}_0)_E = a^4(\nabla^4\chi)_0 \doteqdot \Sigma_4(\chi_I) + 2\Sigma_4(\chi_A) - 8\Sigma_4(\chi_1) + 20\chi_0 = 0, \tag{50}$$

in which $\Sigma_4(\chi_I)$, $\Sigma_4(\chi_A)$, $\Sigma_4(\chi_1)$ stand for the sum of the χ-values at the four symmetrical points typified by I, A, 1, respectively, in figure 5. The effects of a unit increment to χ at a representative nodal point are then exhibited by the square-net pattern shown in figure 6.*

* Cf. Part VIIA, equation (32) and figures 3 and 5*b*.

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For computation within the plastic *enclave*, finite-difference approximations are substituted for θ, ϕ, ψ to obtain a corresponding approximation to (46), written as

$$-(\mathbf{F}_0)_P = \mathbf{c}^2 - \mathbf{a}\theta^2 - \mathbf{b}(\phi^2+\psi^2) = 0 \tag{51}$$

in the notation of 'residuals'. This last relation is made the basis of 'liquidation' when $(\mathbf{F}_0)_P \geqslant 0$, the relation (50) when $(\mathbf{F}_0)_P < 0$.

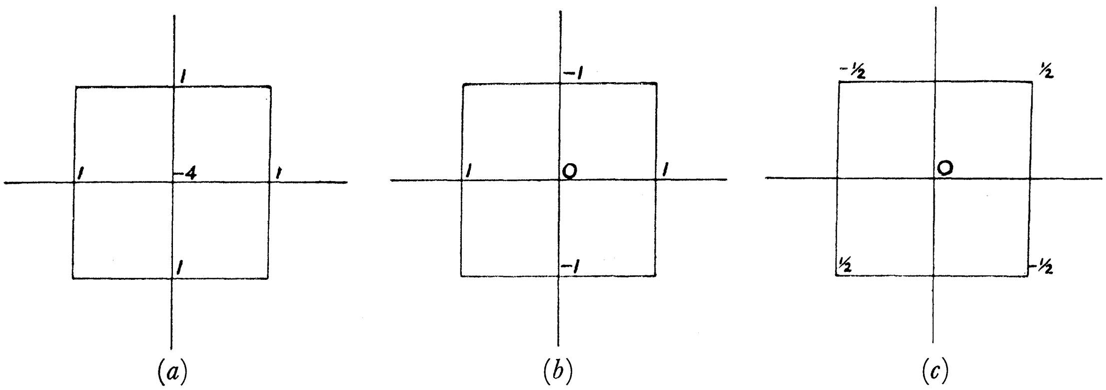

FIGURE 7. Effects on θ, ϕ, ψ of unit increment to χ at a single point (the central node). (*a*) $\delta\theta/a^2$; (*b*) $\delta\phi/a^2$; (*c*) $\delta\psi/a^2$. (a = mesh-side.)

27. Suppose that an infinitesimal increment $\delta\chi_0$ is given to χ at a nodal point 0 within a plastic *enclave*. According to (51), **a**, **b** and **c** being constant,

$$\delta(\mathbf{F}_0)_P = 2\mathbf{a}\theta\,\delta\theta + 2\mathbf{b}(\phi\delta\phi+\psi\delta\psi), \tag{52}$$

and the increments $(\delta\theta, \delta\phi, \delta\psi)/a^2$ resulting from $\delta\chi_0 = 1$ have (in our approximation) values as shown in figure 7 *a*, *b* and *c*. Therefore the 'relaxation pattern' giving the effects of this unit increment to χ_0 is as shown in figure 8.

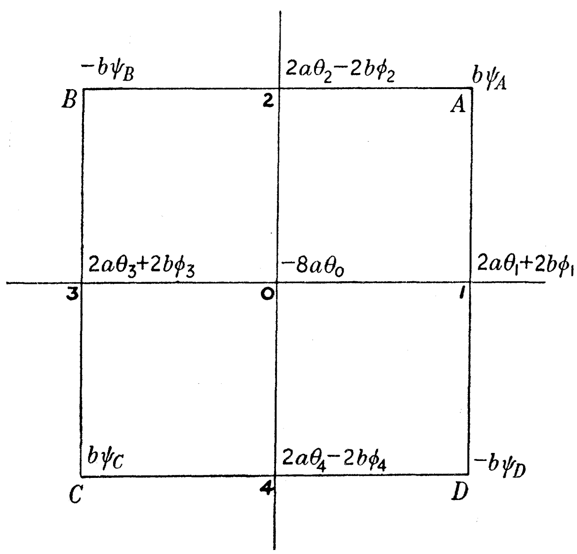

FIGURE 8. (*a* and *b* are the quantities defined in (46)—(48).)

This 'plastic pattern', since it involves local values of θ, ϕ, ψ, differs for different points; moreover it applies, strictly, only to infinitesimal χ-increments. On both accounts the

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relaxation process may be laborious, but in fact (as in earlier papers) the labour was not found prohibitive. *A use of inexact 'patterns' saves time and does no harm provided that exact account is kept of the 'residuals'* (Southwell 1946, §170).

28. Starting from the elastic solution (§23) a normal procedure will bring to zero all **F**'s, as given by (51), which initially are positive; where they are negative the strains are still elastic, so (51) must be replaced by (50), figure 8 by figure 6. Both patterns will be utilized, because changes made in χ within the plastic region will, according to (50), entail 'residuals' at points on the other side of the plastic-elastic interface, and these too must be liquidated.

It has been shown that every stress has continuity across an interface, and it can be deduced that χ, $\partial\chi/\partial x$ and $\partial\chi/\partial y$ are also continuous; so (in our approximation) the ends of any string which cuts an interface are 'real' nodes at which χ has the same values both in the still elastic region and in the plastic *enclave*. A unique solution is to be expected, since one or other of (26) and (51) must be satisfied at every point where χ may be varied.

According to (52) and figure 7, the effect of a unit increment $\delta\chi_0$ upon the total of the 'residuals' at adjacent points is the finite-difference approximation to

$$2a^2\left[\mathbf{a}(\nabla^2\theta)+\mathbf{b}\left\{\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\phi+2\frac{\partial^2\psi}{\partial x\,\partial y}\right\}\right]_0 = 2a^2(\mathbf{a}+\mathbf{b})\,(\nabla^4\chi)_0, \quad \text{according to (44).} \tag{53}$$

It is therefore zero in the initial (elastic) solution, where χ is biharmonic. On this account the positive residuals given by (51) will at first be transferred without alteration of their total, and it is to be expected that an *enclave* will extend rather widely before liquidation is complete.

The computation of displacements in an enclave

29. Having the stresses (both incremental and total) we can proceed to compute displacements and components of strain by methods (*not* entailing relaxation) which have been outlined in §§19–22.

III. Computational results. Conclusion

30. The two examples of §23 really constitute four, since each was treated under conditions both of plane strain and of plane stress. The work was carried to finer detail in Example 1, for which the finest net employed in computation was twice as fine as the net which is shown in figures 10 and 11. The numbers computed for the sharp-notched specimen (Example 2: figures 23 to 34) may conceivably include small errors; but their general correctness may be accepted.

The numerous diagrams are considered all to be necessary for a complete presentation of the four solutions. (The space required for comparable tables would be prohibitive.)

Results for Example 1

31. First, conditions of *plane strain* were postulated, so that **a**, **b**, **c** in (46) and figure 8 had the values given in (48). In Example 1 the stress-component X_x is everywhere large in comparison with Y_y and X_y, so that approximately

$$\phi = -(X_x - Y_y) = -Q, \quad \psi = -2X_y = 0. \tag{54}$$

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Consequently the simpler 'pattern' shown in figure 9 could be employed—with advantage in respect of labour—during the early stages of liquidation. The exact expression (51) had, of course, to be used for computation of residuals; but these could be liquidated *completely* with a use of figure 9, partly because (54) were close approximations, but also because the required alterations to Q were small of the order of 2 %.

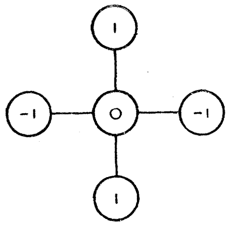
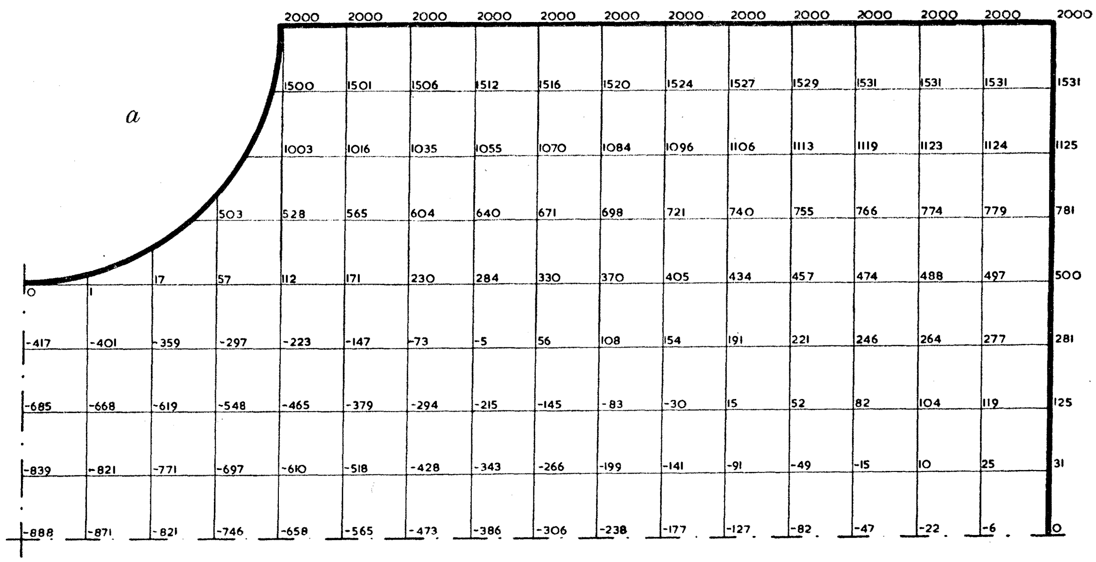
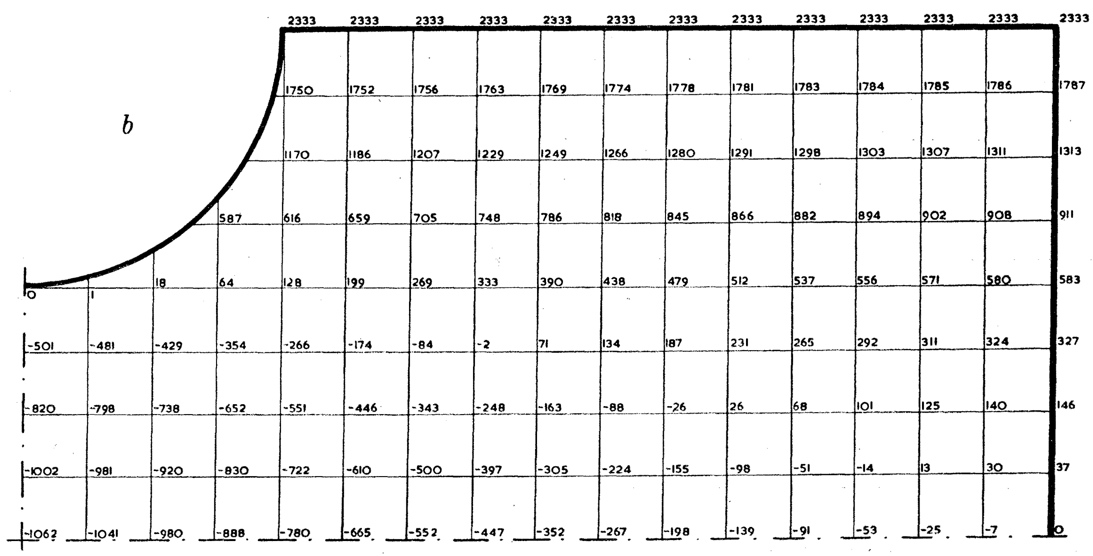


FIGURE 9. Approximate 'relaxation pattern' for increments $\delta Q/a^2$ due to unit increment $\delta\chi$ at the central node.





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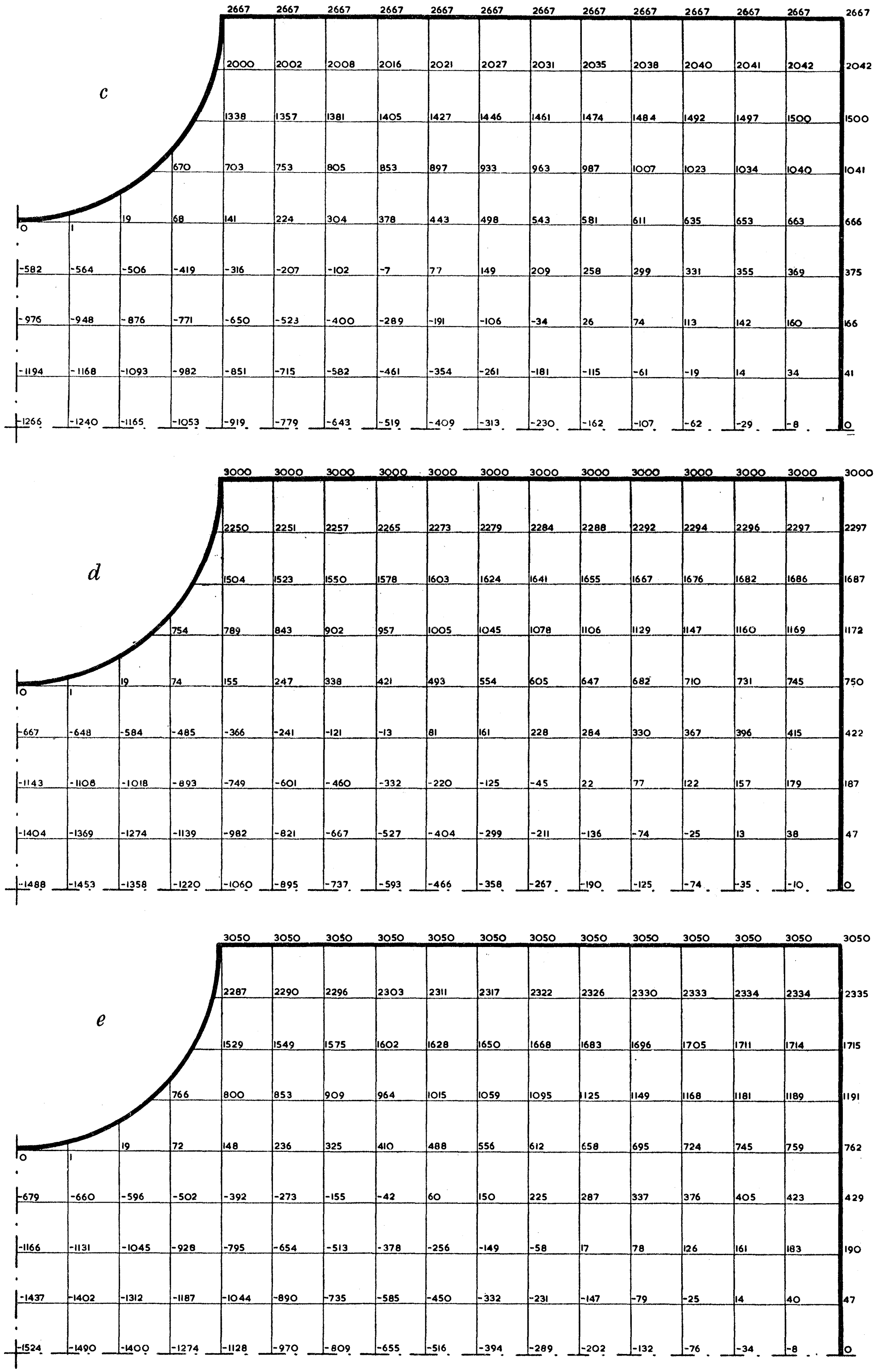

FIGURE 10. Stress-function χ in five cases of plastic straining. (Example 1: plane strain. The lettering *a* to *e* is explained in figure 13.)

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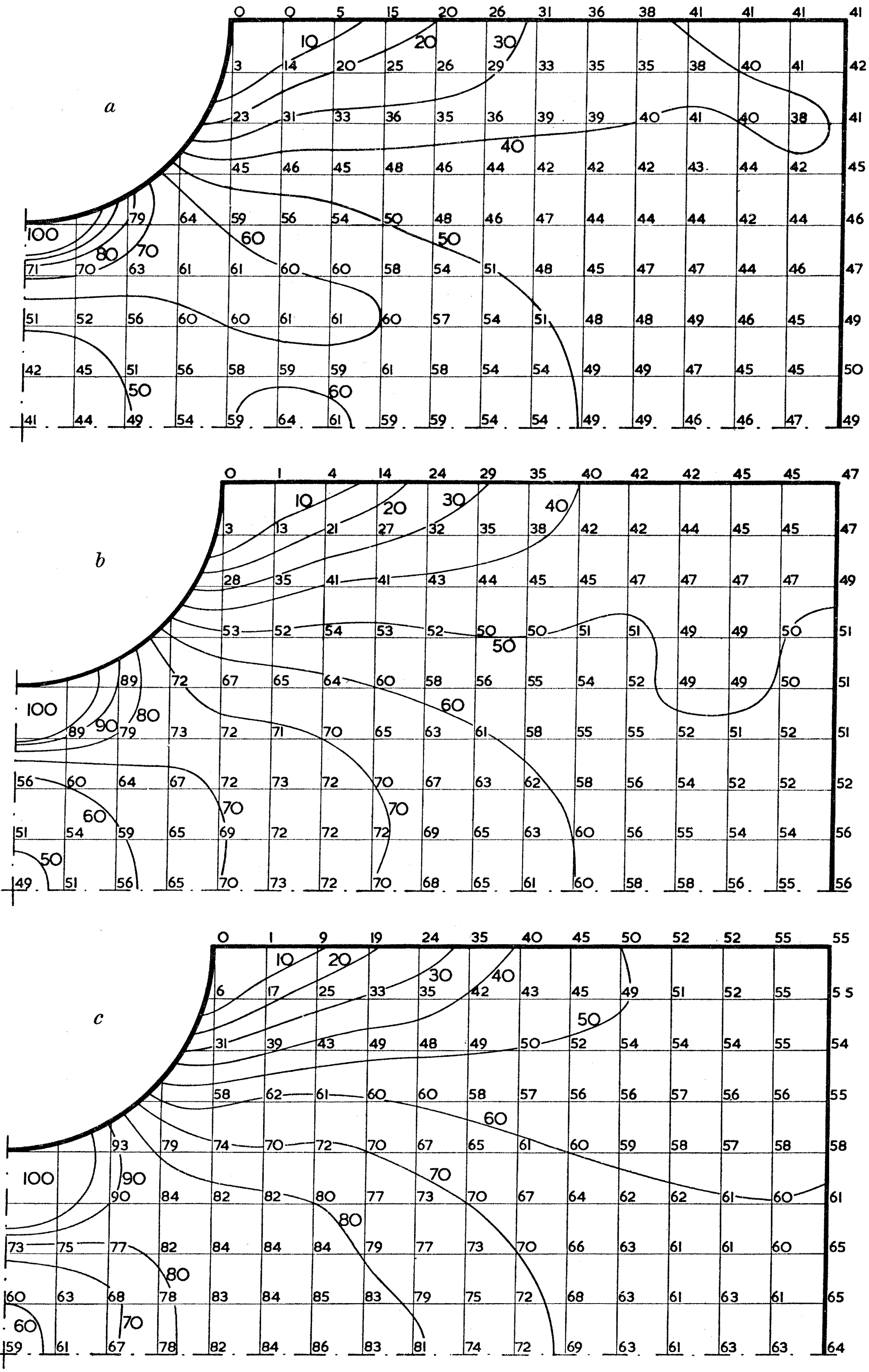

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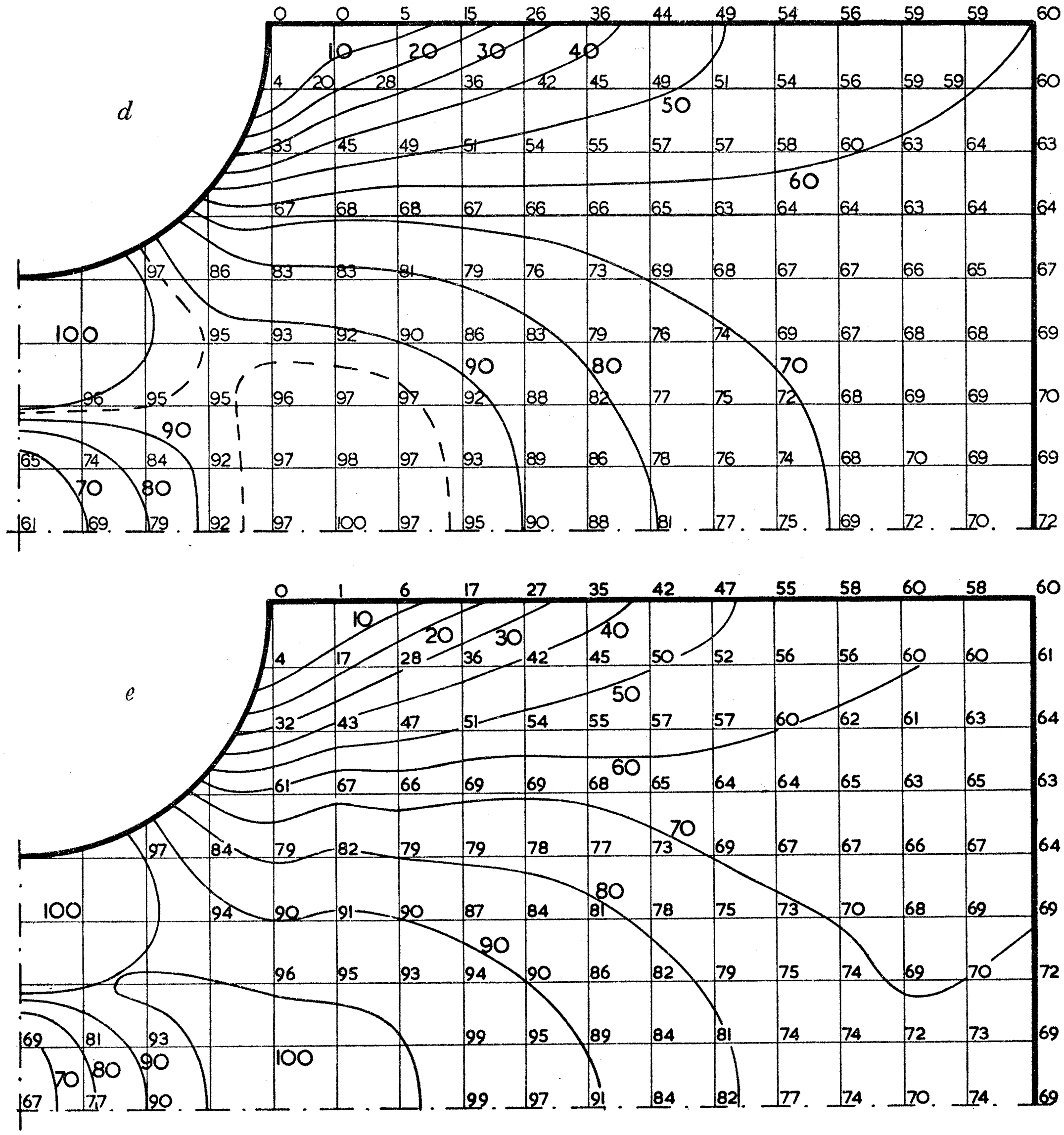

FIGURE 11. 'Mises-Hencky equivalent stress' as percentage of its maximum value. (Example 1: plane strain. For the lettering *a* to *e*, see figure 13.)

For the same reason the *enclave* extended rapidly, confirming the anticipation of §28. T_L denoting the resultant tension at which plastic straining first began, another *enclave* started to form when the tension attained a value of $1{\cdot}81T_L$, and at a value $1{\cdot}84T_L$ the regions coalesced so as to extend right across the specimen.

32. Figure 10 records computed values of χ for five values of the applied tension. Figure 11 (derived from them) exhibits corresponding values and contours of the quantity $Q/\mathbf{c}$ defined in (46), which of course has a value 100% throughout the *enclave*. In figure 12 the *enclave* and the immediately adjoining region were drawn to a larger scale; the finest net employed in computation (cf. §30) is shown, and values of θ_1, the first root of equation (14), are recorded (the second root $\theta_2 = \frac{1}{2}\pi + \theta_1$). Figure 13 summarizes the foregoing diagrams, showing the growth of the *enclave* with increasing tension.

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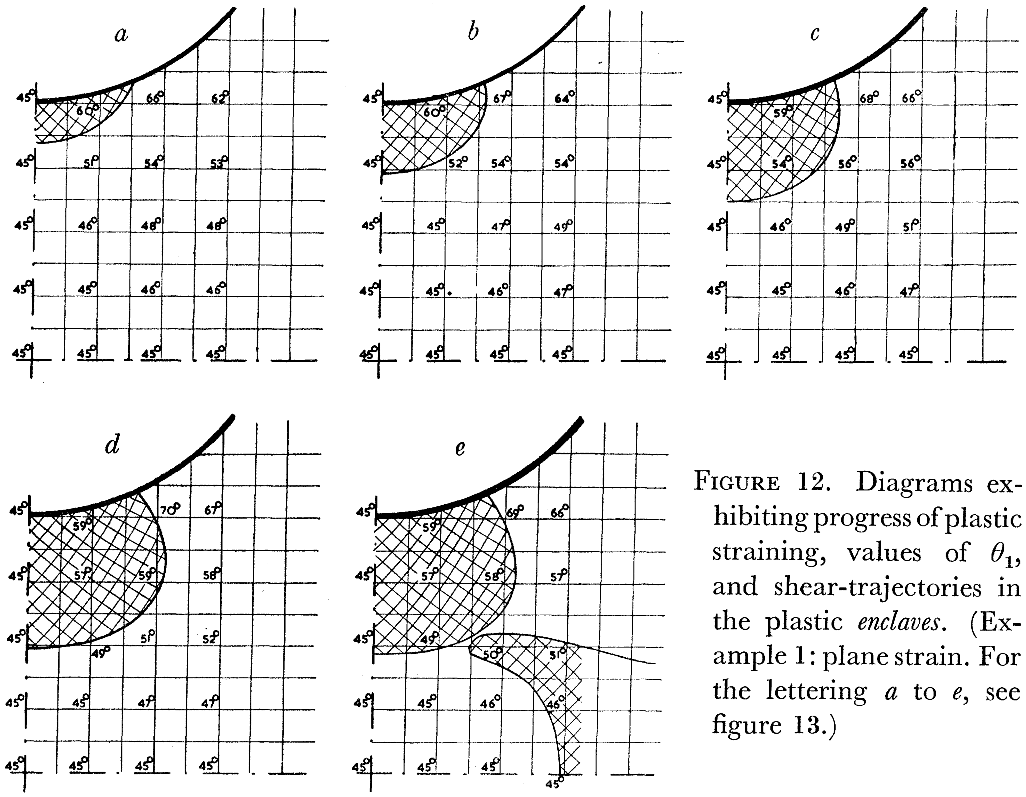

Figure 12. Diagrams exhibiting progress of plastic straining, values of θ_1, and shear-trajectories in the plastic *enclaves*. (Example 1: plane strain. For the lettering *a* to *e*, see figure 13.)

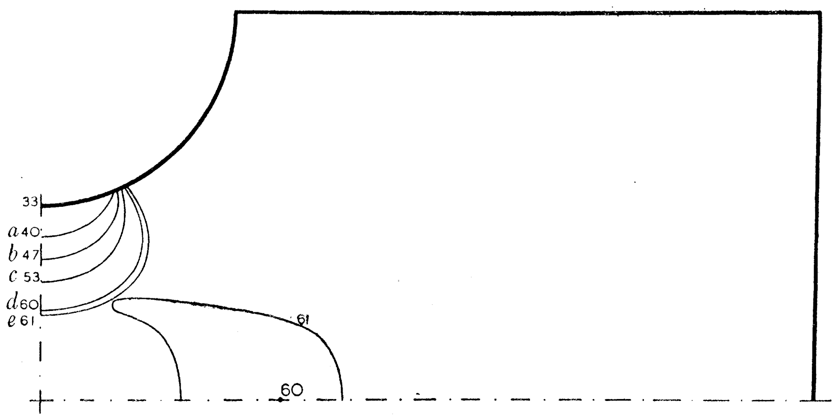

Figure 13. Extension of the boundaries of the plastic *enclaves*. (Example 1: plane strain.)

Figure 14 records values of the displacements for conditions of plane strain and indicates the resulting distortion of the semicircular notch. Figure 15 shows the influence of increasing tension on the contraction of the 'waist'. (In these diagrams the measure of the applied tension is T, the uniform tensile stress at the ends of the specimen, expressed as a percentage of the limiting 'Mises-Hencky equivalent stress', and the displacements u and v are expressed as multiples of $kL/(200\surd 3E)$, L denoting the width of the specimen across its 'waist'.)

33. Similar treatment gave the results for *plane stress* which are recorded in figures 17 to 22 (the correspondence of those diagrams with figures 10 to 15 will be apparent from their legends). **a**, **b**, **c** had values, taken from (47), in which the ratio **a**/**b** is no longer small; and in consequence, although the approximations (54) were valid, the 'pattern' (figure 16) was less simple than figure 9. Computed displacements are shown in figure 21 (u and v

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being now expressed as multiples of $KL/(400\ E)$, where L as before denotes the waist-width). The results are generally similar (though figures 12 and 19 differ sensibly) to those found previously for conditions of plane strain.

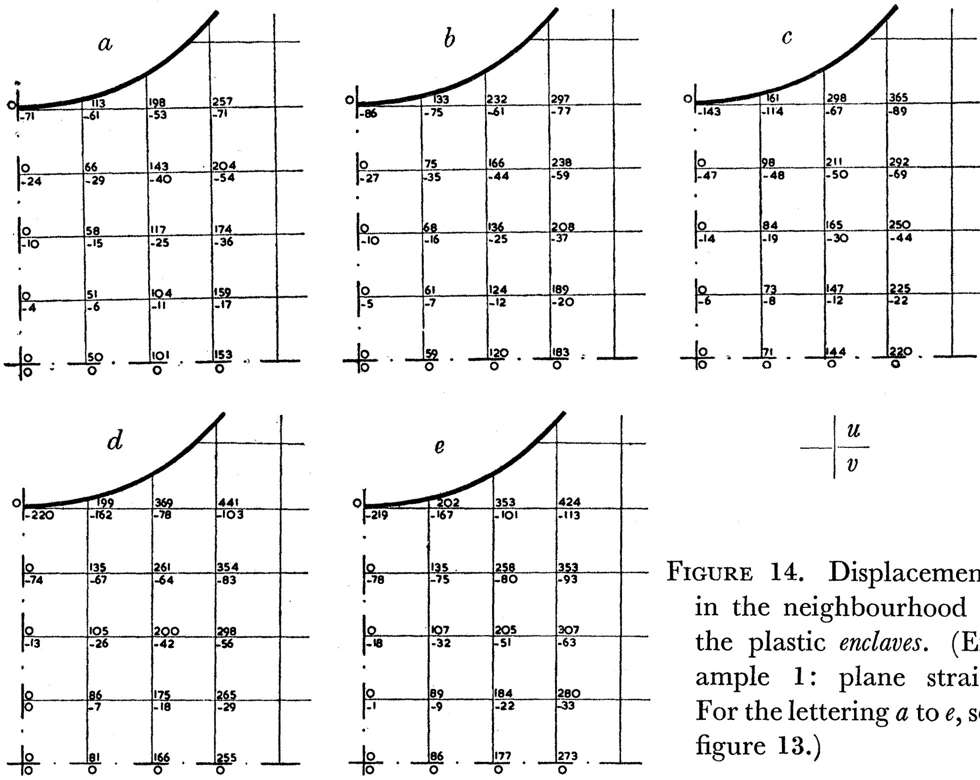

FIGURE 14. Displacements in the neighbourhood of the plastic *enclaves*. (Example 1: plane strain. For the lettering *a* to *e*, see figure 13.)

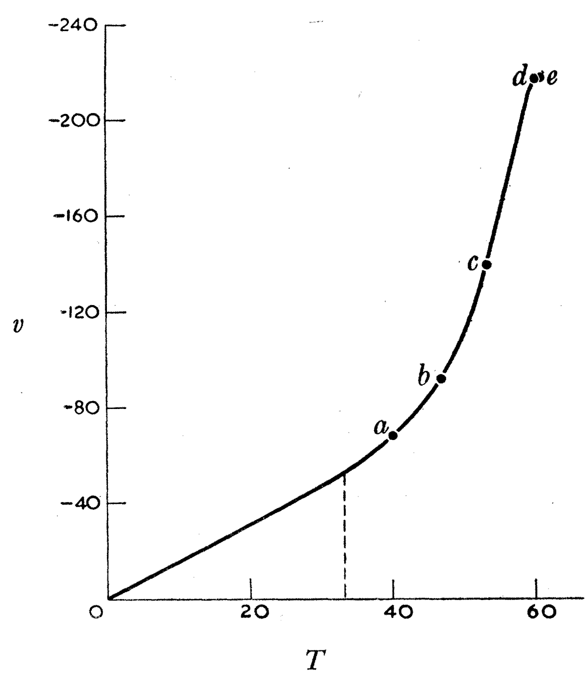

FIGURE 15. Progress of contraction at the 'waist'. (Example 1: plane strain. The total contraction is $-2v$; T is a measure of the applied tension. Cf. §32.)

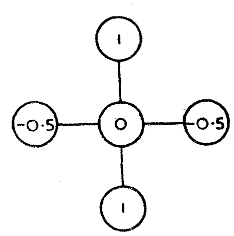

FIGURE 16

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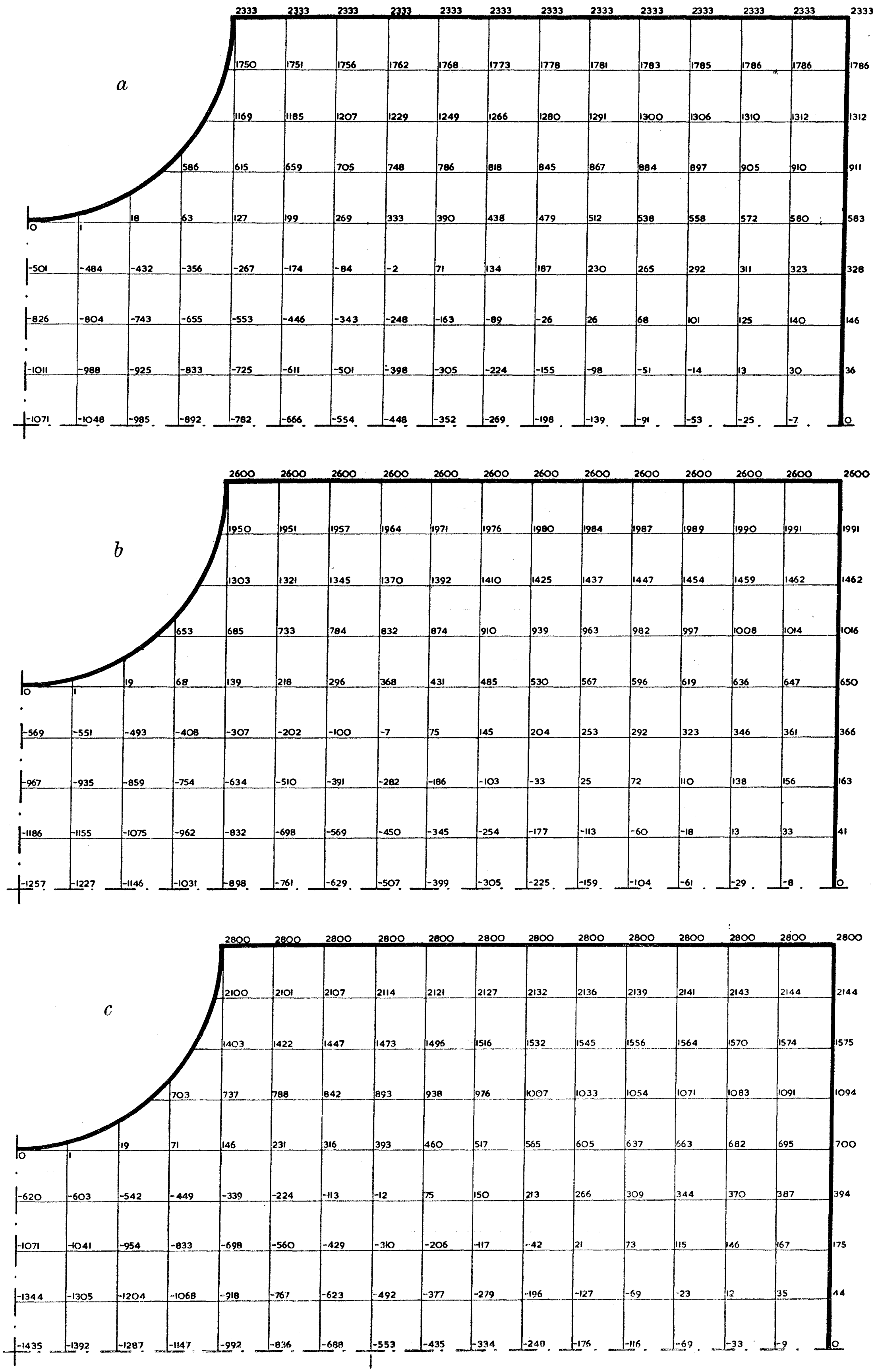

Figure 17. Stress-function χ in three cases of plastic straining. (Example 1: plane stress. The lettering a to c is explained in figure 20.)

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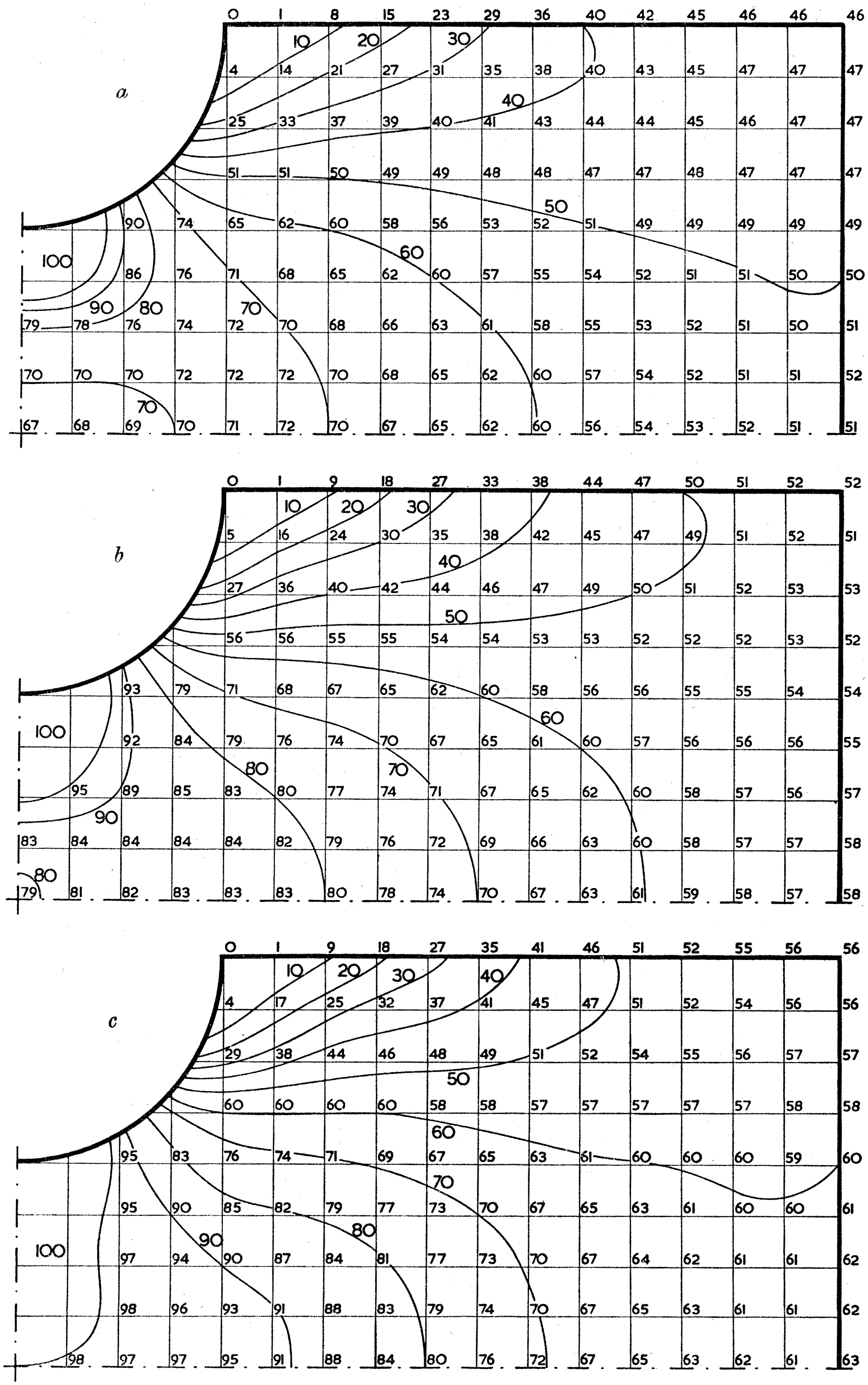

FIGURE 18. 'Mises-Hencky equivalent stress' as percentage of its maximum value. (Example 1: plane stress. For the lettering *a* to *c*, see figure 20.)

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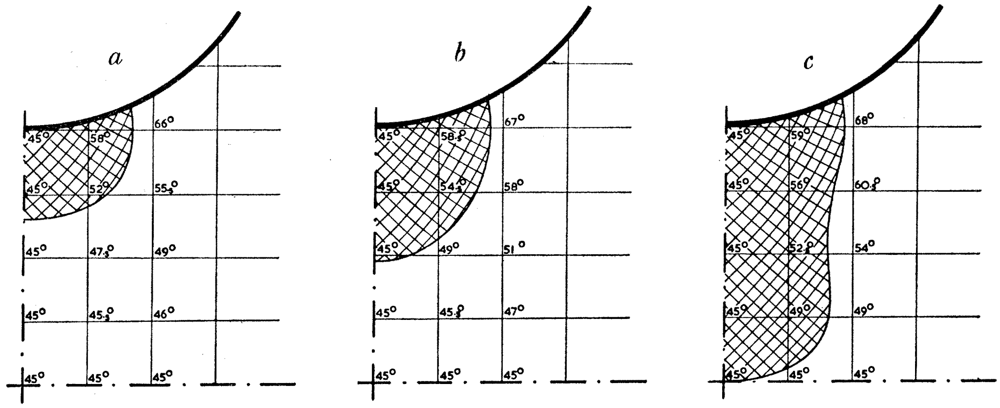

FIGURE 19. Diagrams exhibiting progress of plastic straining and shear-trajectories in the plastic *enclaves*. (Example 1: plane stress. For the lettering *a* to *c*, see figure 20.)

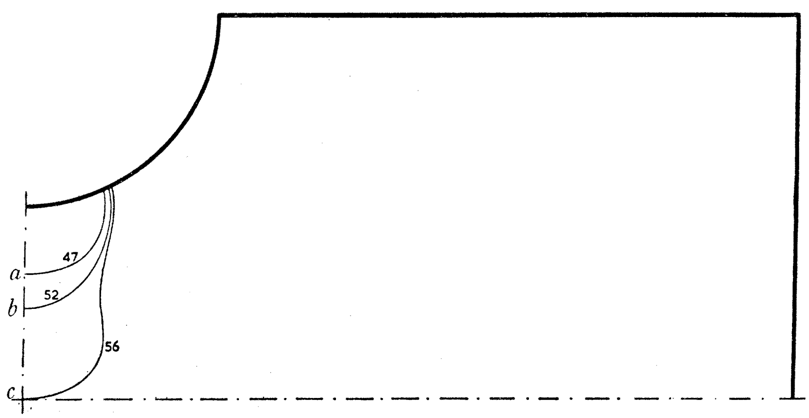

FIGURE 20. Extension of the boundaries of the plastic *enclaves*. (Example 1: plane stress.)

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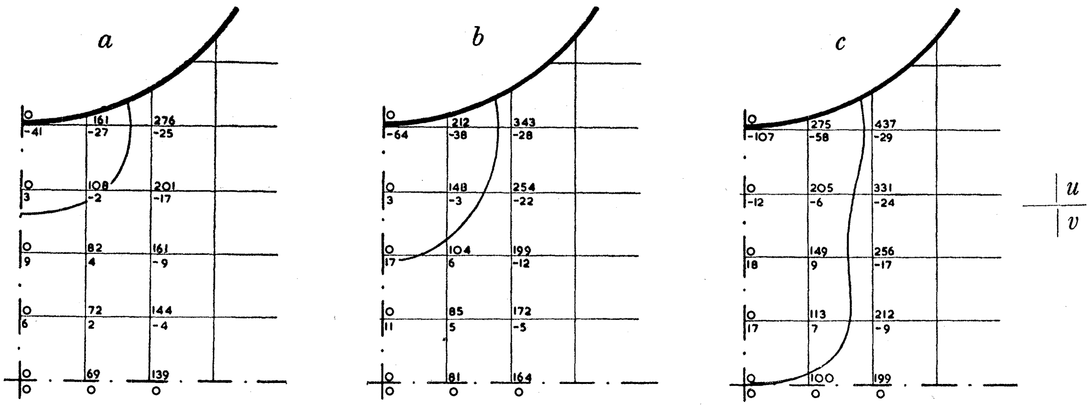

FIGURE 21. Displacements in the neighbourhood of the plastic *enclaves*. (Example 1: plane stress. For the lettering *a* to *c* see figure 20.)

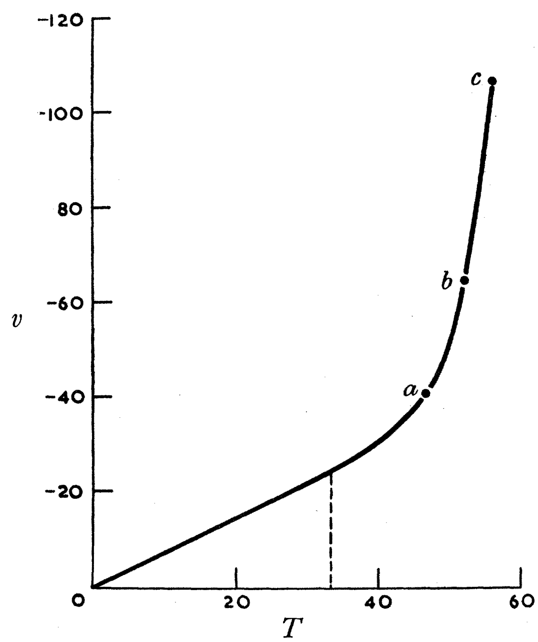

FIGURE 22. Progress of contraction at the 'waist'. (Example 1: plane stress. T is a measure of the applied tension (cf. § 32). The total contraction is $-2v$.)

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Results for Example 2

34. For this example a coarser 'net' was used (§30), our interest being less concerned with computational problems than with the general nature of the plastic straining. Figures 26 and 32, which exhibit the growth of the plastic *enclaves*, show that in this example the occurrences are widely different. Figures 28 and 34 show the 'waist contraction' to be relatively small.

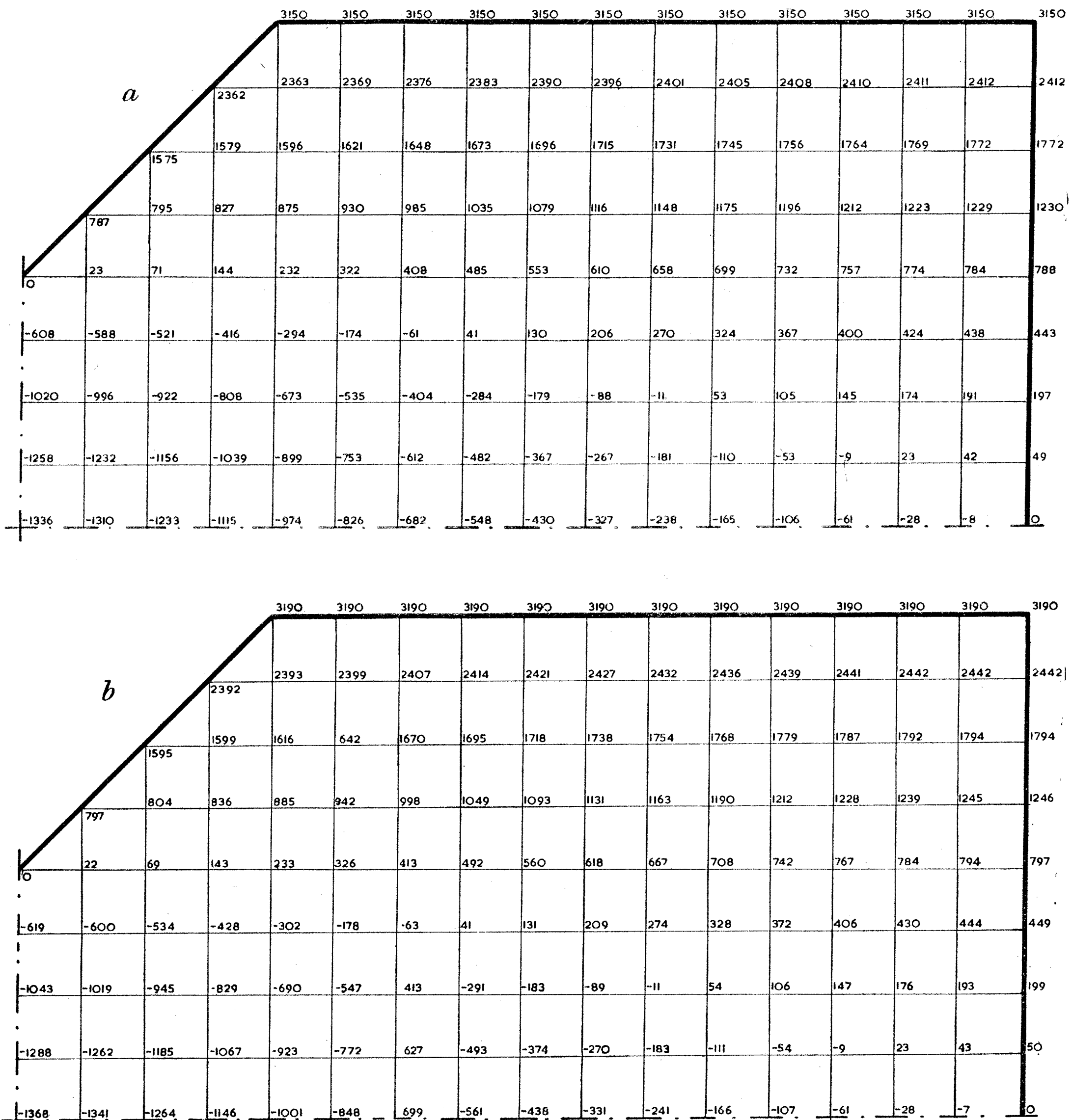

FIGURE 23. Stress-function χ in two cases of plastic straining. (Example 2: plane strain. The lettering *a* and *b* is explained in figure 26.)

Again a considerable number of diagrams is reproduced in order that all relevant computations may be recorded. (In this example, too, our treatment covered both plane strain and plane stress, and here the differences between the two cases are more striking (cf. figures 25 and 31). For the units in which T, u and v are measured, cf. §§32 and 33.)

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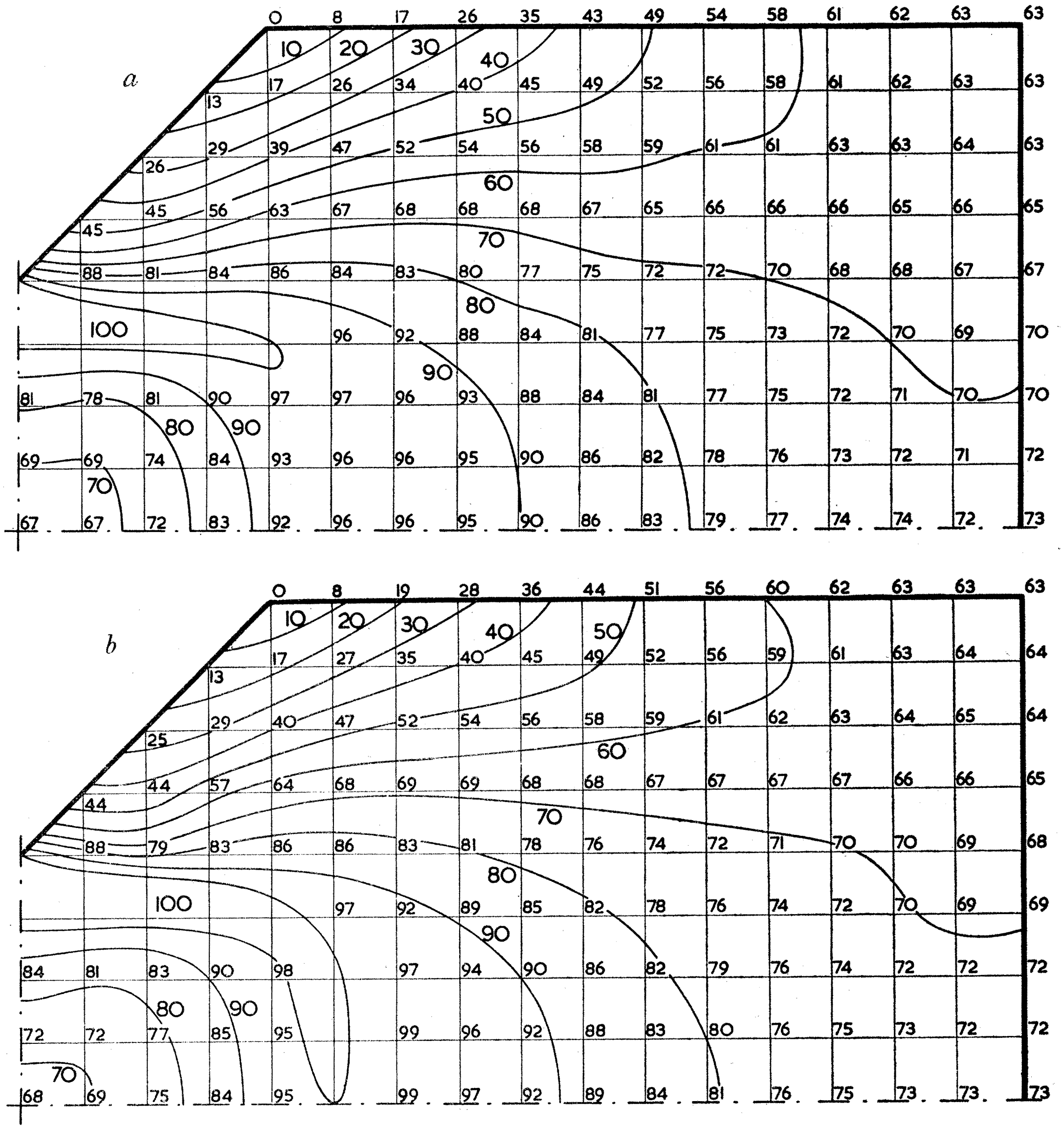

FIGURE 24. 'Mises-Hencky equivalent stress' as percentage of its maximum value. (Example 2: plane strain. For the lettering *a* and *b*, see figure 26.)

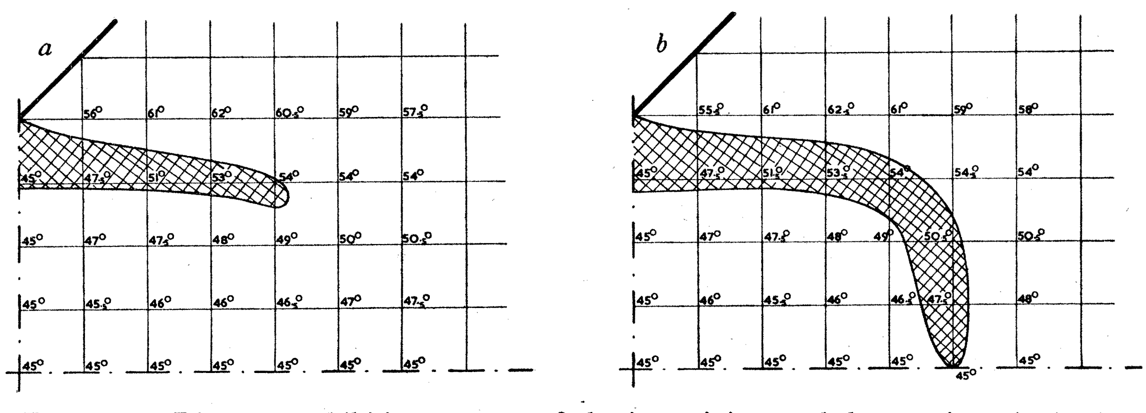

FIGURE 25. Diagrams exhibiting progress of plastic straining, and shear-trajectories in the plastic *enclaves*. (Example 2: plane strain. For the lettering *a* and *b*, see figure 26.)

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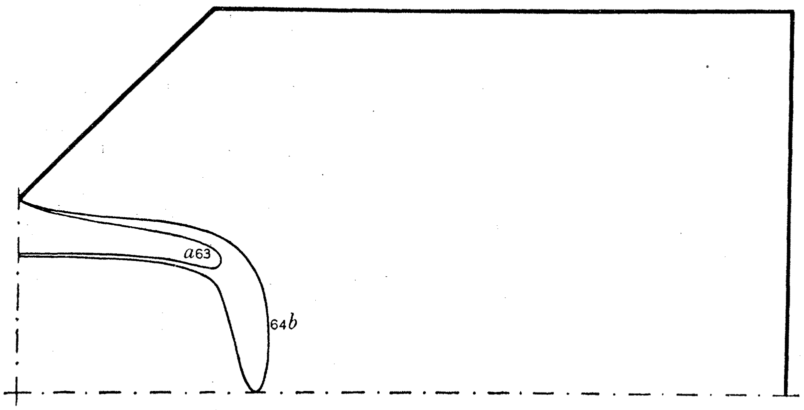

FIGURE 26. Extension of the boundaries of the plastic *enclaves*. (Example 2: plane strain.)

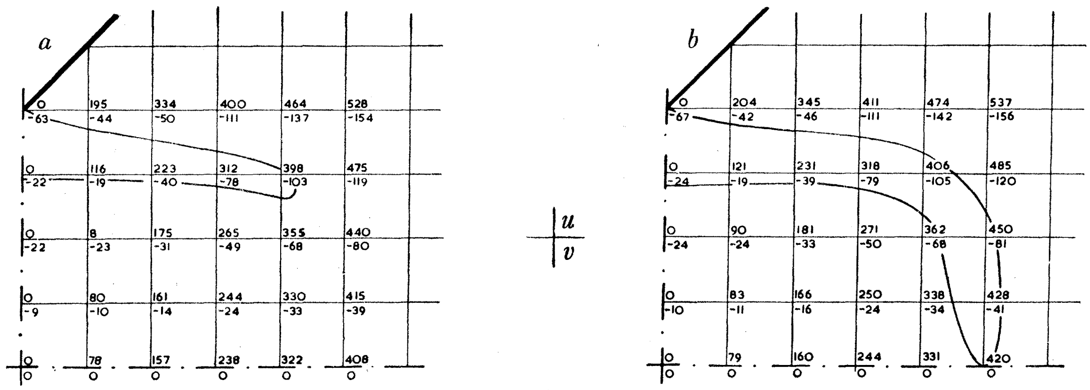

FIGURE 27. Displacements in the neighbourhood of the plastic *enclaves*. (Example 2: plane strain. For the lettering *a* and *b*, see figure 26.)

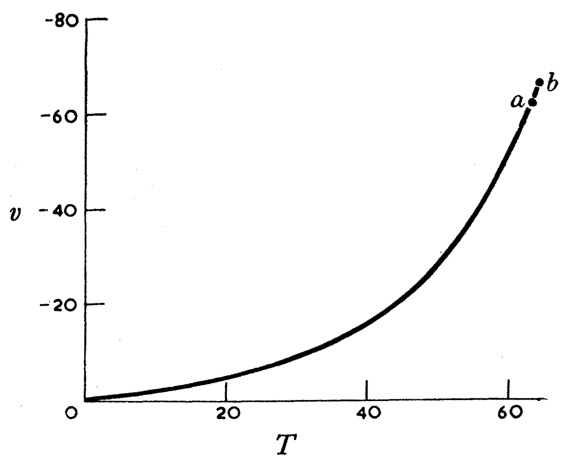

FIGURE 28. Progress of contraction at the 'waist'. (Example 2: plane strain. T is a measure of the applied tension (cf. § 32). The total contraction is $-2v$.)

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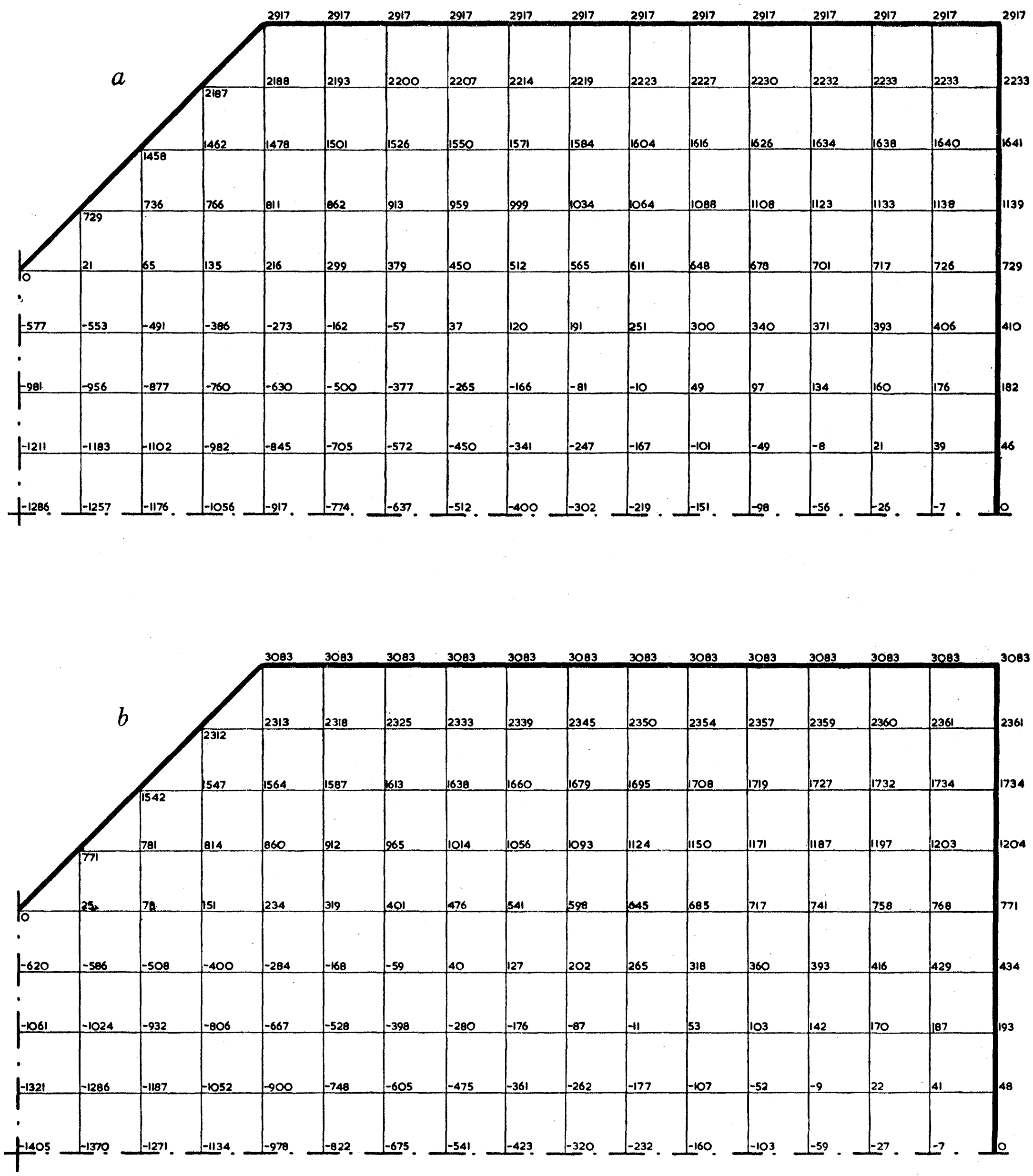

FIGURE 29. Stress-function χ in two cases of plastic straining. (Example 2: plane stress. The lettering a and b is explained in figure 32.)

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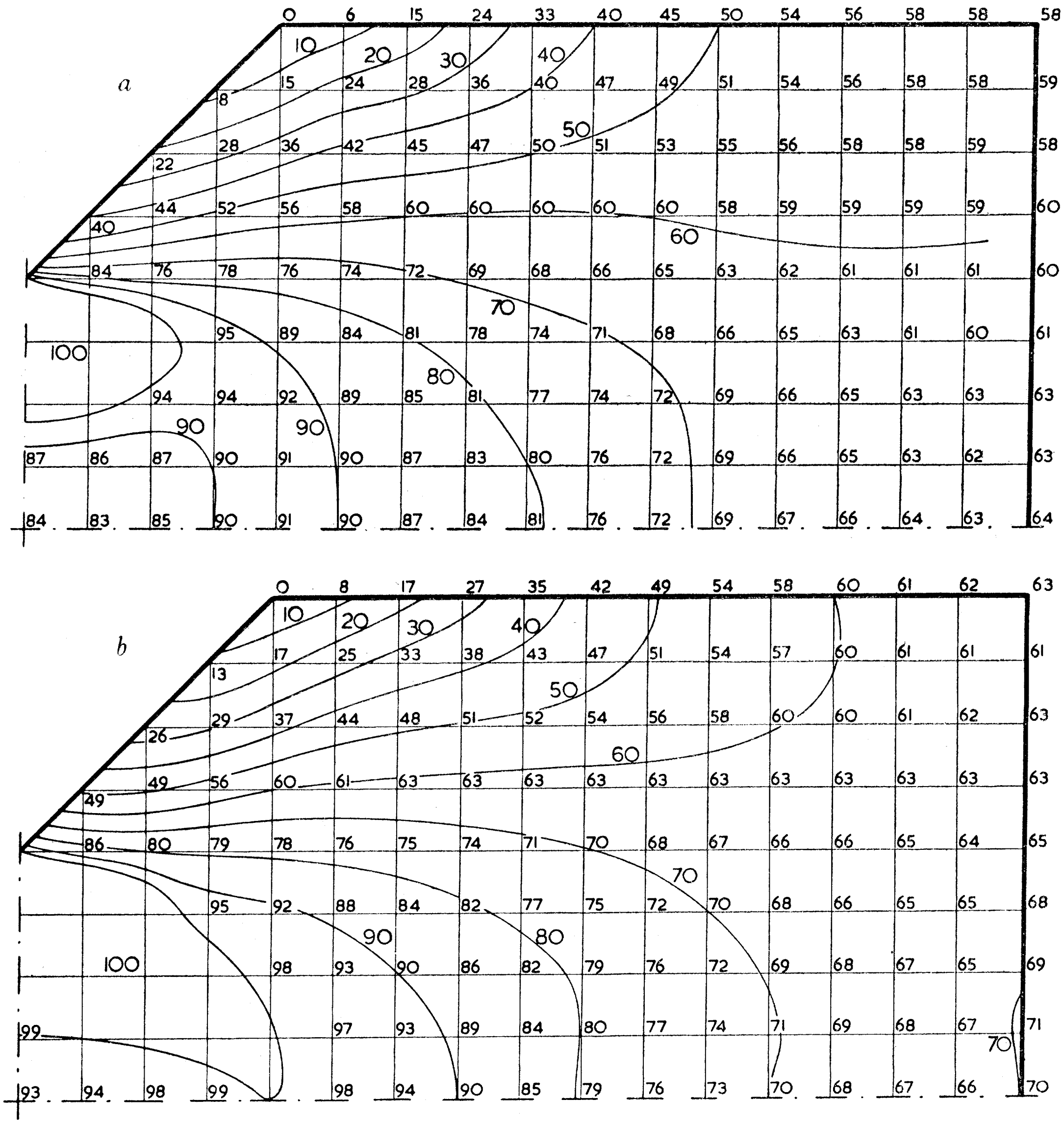

Figure 30. 'Mises-Hencky equivalent stress' as percentage of its maximum value. (Example 2: plane stress. For the lettering *a* and *b*, see figure 32.)

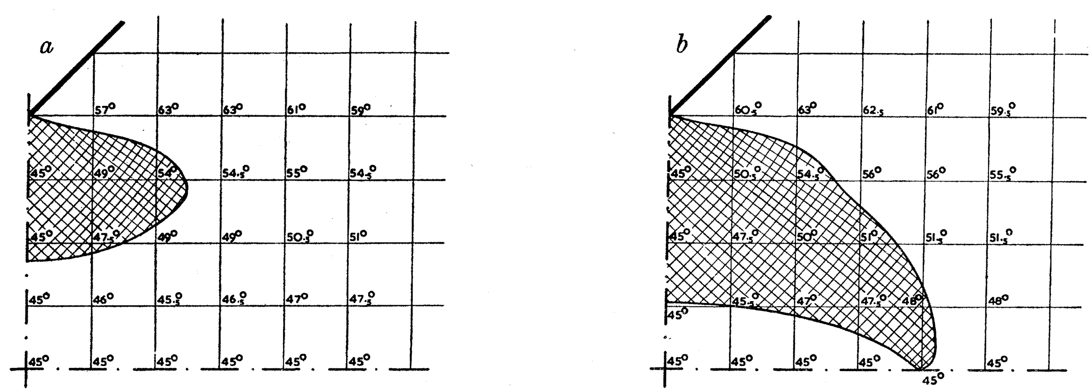

Figure 31. Diagrams exhibiting progress of plastic straining and shear-trajectories in the plastic *enclaves*. (Example 2: plane stress. For the lettering *a* and *b*, see figure 32.)

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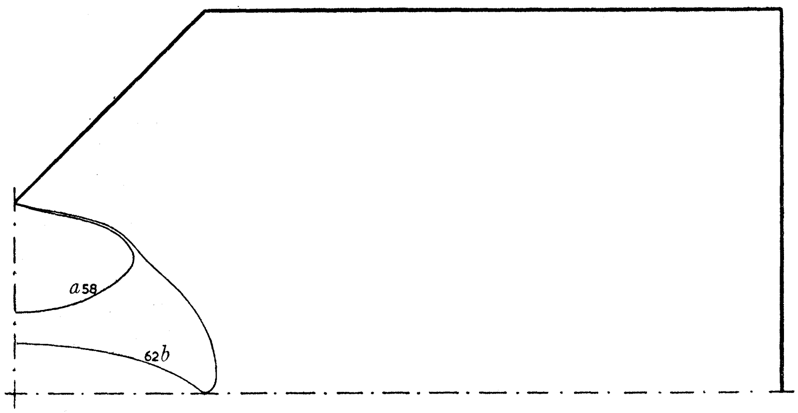

FIGURE 32. Extension of the boundaries of the plastic *enclaves*. (Example 2: plane stress.)

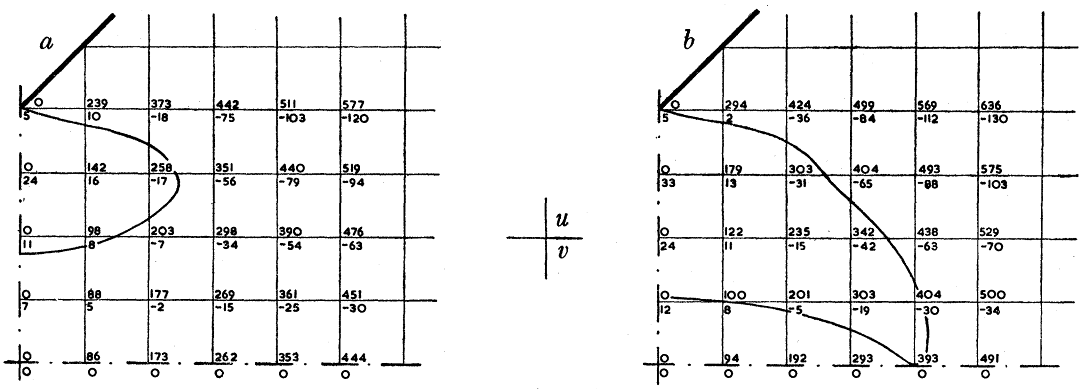

FIGURE 33. Displacements in the neighbourhood of the plastic *enclaves*. (Example 2: plane stress. For the lettering *a* and *b*, see figure 32.)

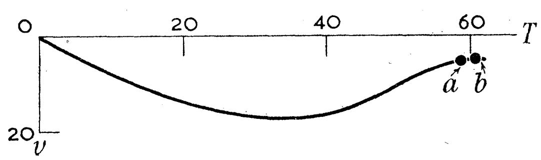

FIGURE 34. Progress of contraction at the 'waist'. (Example 2: plane stress. T is a measure of the applied tension (cf. §32). The total contraction is $-2v$.)

CONCLUSIONS

35. There would seem to be no doubt that Relaxation Methods can deal with problems of plastic straining in two dimensions under any likely assumption regarding its criterion, and there also seems to be a reasonable prospect of extending them to the harder problem of stress-systems having axial symmetry. That prospect will be explored later, also the practicability of computing 'residual stresses' in overstrained material. (This last is almost certain; for assuming all the material to remain elastic during an unloading process we can calculate the reversed stresses thereby induced, and superposing these on the stresses computed previously we have the stresses that will persist *unless plastic strain occurs in the process of unloading*. It will do so, if in any region these computed stresses violate the Mises-Hencky criterion: in that event adjustment will have to be made, by the methods of this paper, both in the stresses and in the strains.)

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36. Pending these further studies, a tentative deduction may be drawn from figures 12, 19, 25 and 31. In either type of specimen, when the *enclaves* have extended from the two notches so as to meet, failure could result solely from plastic slipping, *but the surface on which such slipping must occur is fairly lengthy*—it cannot 'take a short cut' across the waist. Now failure can also occur by a breakdown of cohesion—direct separation under tensile stress; and such separation is most likely to take place across the *shortest* surface—that is, across the still elastic portion of the waist. It is thus to be expected that the actual mode of fracture will entail plastic slipping in the neighbourhood of the notch boundary, tensile fracture across the central portion of the waist; so our results go some way towards explaining the type of fracture known as 'cup and cone'.

Résumé of assumptions

37. The physical hypotheses which are the basis of our treatment have been detailed in §§3 to 6. Here it may be useful to bring together other assumptions which were found necessary in the course of our work. Some are not strictly justifiable; others are justified in respect of the examples treated, namely:

(§10) that the boundary is simply-connected;

(§§12, 17) that the plastic-elastic interface is neither straight nor circular, and that every component displacement is continuous;

(§14) that $\Delta\lambda = 0$ at those points (if any) where our proof of its evanescence fails because $X_x = Y_y = Z_z$;

(§19) that the strains are everywhere small enough to justify use of their normal expressions in terms of displacement.

Assumptions judged defensible only as making for simplicity without entailing likelihood of serious error are:

(§§9, 15) the assumptions made in regard to the principal stress Z_z (which lead to a slightly inexact criterion in our treatment of plane strain);

(§11) the use of finite (and fairly large) increments of load in the step-by-step study of plastic strain.

To these should be added the assumption (made in all papers of this series) that derivatives may be replaced by finite differences. The consequent error depends upon the mesh-size of the 'ultimate relaxation net'. It can hardly be large in the solutions of this paper.

Appendix

Strains entailed by the solution for plastic torsion (§§2 *and* 4)

Strains in relation to the torsion problem were not considered in Part III; but the stresses there deduced by means of the 'Prandtl roof' (§2) have fixed directions *once the straining has become plastic*, so can be related with the *total* strains. On the assumption that all of these are zero except e_{zx} and e_{zy}, it follows from (8) that

$$e_{zx}/X_z = e_{yz}/Y_z = \frac{1}{\mu} + 3\lambda = k \text{ (say)}, \qquad \text{(i)}$$

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μ (the modulus of rigidity) relating to the elastic part, and 3λ (an undetermined quantity) to the plastic part of the strain. $\lambda = 0$ when the material is still elastic, and in regions of plastic straining it can have any positive value.

In terms of Ψ, the 'stress function'—by definition (Love 1927, §219)—

$$X_z = \mu\tau \frac{\partial \Psi}{\partial y}, \quad Y_z = -\mu\tau \frac{\partial \Psi}{\partial x}, \tag{ii}$$

and so by (i)

$$\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = e_{zx} = k\mu\tau \frac{\partial \Psi}{\partial y}, \quad \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = e_{zy} = -k\mu\tau \frac{\partial \Psi}{\partial x} \tag{iii}$$

everywhere, $k\mu$ $(= 1+3\lambda\mu)$ denoting an undetermined quantity which, with Ψ, is independent of z.

Now $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial w}{\partial z} = 0$ by assumption. Hence and from (iii), if rigid-body displacements are disregarded,

$$u = -\tau yz, \quad v = \tau zx, \tag{iv}$$

and

$$\frac{\partial w}{\partial x} = k\mu\tau \frac{\partial \Psi}{\partial y} + \tau y, \quad \frac{\partial w}{\partial y} = -\left(k\mu\tau \frac{\partial \Psi}{\partial x} + \tau x\right), \tag{v}$$

where τ (as before) denotes the constant twist per unit length.

Then for compatibility of the expressions (v)

$$\frac{\partial}{\partial x}\left(k\mu \frac{\partial \Psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\mu \frac{\partial \Psi}{\partial y}\right) + 2 = 0, \tag{vi}$$

and this condition is satisfied by Ψ as found in the manner of Part III. For where the strain is elastic $k\mu = 1$ and Ψ is governed by

$$\nabla^2 \Psi + 2 = 0,$$

so (vi) is already satisfied; and in plastic regions Ψ (being determined by the 'Prandtl roof') is a known function of x and y, so (vi) is an equation which determines $k\mu$ $(= 1+3\lambda\mu)$.

A 'Prandtl roof' is generated by a line of constant slope $(f_Y/\mu\tau)$ which in plan is a radius of curvature of the boundary. If this is taken as the axis Ox and O as lying at the centre of curvature, then for any point on Ox

$$-\frac{\partial \Psi}{\partial x} = f_Y/\mu\tau, \quad \frac{\partial^2 \Psi}{\partial x^2} = 0, \quad \frac{\partial \Psi}{\partial y} = 0, \quad \nabla^2 \Psi = \frac{1}{x}\frac{\partial \Psi}{\partial x},$$

and in consequence, after expansion, equation (vi) reduces to

$$\left[\frac{1}{x} + \frac{\partial}{\partial x}\right](k\mu) = 2\mu\tau/f_Y.$$

This has the general solution

$$k\mu = \mu\tau x/f_Y + A/x,$$

the constant A being fixed by the condition that $k\mu = 1$ at the plastic-elastic interface when $x = x_0$ (say). The final solution (giving the variation of $k\mu$ along Ox) is

$$3\lambda\mu + 1 = k\mu = [(\mu\tau/f_Y)(x^2 - x_0^2) + x_0]/x.$$

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